THE MATHEMATICS STUDENT

Volume XI] MARCH & JUNE 1943 [Nos. 1 & 2

A DESIGN FOR A PERPETUAL CIVIL CALENDAR

BY

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This design given on pages 4 and 5 will give the calendar of any month past or future reckoned according to the Gregorian system, now followed in most of the countries. This was adopted in England and her colonies from the 14th of September 1752. According to this, a century year is to be considered a leap year only if its century number is divisible by four.

The week days are written down in cyclical order in 7 rows so that they show all possible settings for the calendar of a month beginning from the 1st of that month.

Table I consists of the century numbers (19 in 1943) beginning from 1800, and the number of the year in the century (43 in 1943) is set down in table II. Together they fix any year of any century.

Part of table III consists of the cycle of seven letters M S E H N O Y beginning in each column with different letters, but keeping the cyclic order. Take the first column of this table and pick out the letter H corresponding to 18 of table I and 00 of table II. Let this correspond to any date of 1800; then the same date for 1801 will occur on the next week day, being the 366th day, and so against 18 of I and 01 of II comes the next letter N in the cycle of 7 letters. Similarly for 1802 and 1803 there is a shift of 1 letter each. But when we pass on from 1803 to 1804, since the latter is a leap year of 29 days for February there will be a shift of two days from March 1st of 1803 to the corresponding date of 1804; and so, 04 is put against S and not against M. Now this letter should be in the previous row for January of a leap year, as could be seen from the place of S in the 1st row and fifth column when January of a leap year is taken. The same thing holds good for February also. If O is

the letter for February of a leap year, then as the leap year reckoning affects only after February 28 the shift for a date in February of a leap year from the corresponding date of previous year is only one date and so O is put one row above for February (leap) different from O for other months of a leap year, or February for ordinary years.

Proceeding in this manner we come to 1899 with the letter M corresponding to it. Now the week day corresponding to any date of 1900 is the one next to that for 1899 as 1900 is not a leap year. So bring the next letter S in column 2 against 19 of table I and 00 of table II, and fill up the 2nd column with the seven letters keeping the same cyclic order as in previous column beginning from S against '00'. Proceeding through all the years of the 20th century one comes to O for 1999, and the letter for 2000 being a leap year is thus M. So we place this letter M against 20 of I and '00' of II. Then we get by keeping the cyclic order all letters for the 3rd column, and get N as the letter for 2099 giving O for 2100; thus insert O against 21 of I and '00' of II; completing the cycle of this column we get for 2199, the letter E, giving for 2200 "H" which is what we had for 1800. So we write down 22 under 18, as the cycle of letters of column 1 of III will hold good for the whole of this century; again for 23 the 2nd column will hold good and for 24 the 3rd, and for 25 the 4th. Thus there are only 4 cycles of seven representative letters possible for all the years, and the order gets repeated after four centuries.

Now consider the distribution of the months giving all possible week days for their beginnings.

We have written down the week days in seven possible rows corresponding to the seven letters M S E H N O Y of the first column of Table III. Any one of the rows may give the week days for a month beginning from the 1st, and any letter for a year occurs in one of seven rows in III. Let the letter M of the first column in III give the calendar for all months of a year say (1899). January of 1899 began with Sunday. For January and October of an ordinary year differing by 39 weeks it will be the same calendar. Hence October is put down along with January in the 1st column; The first of February of the same year will be a Wednesday, as the 1st January is a Sunday, and so February is put in the column of M in the row beginning with Wednesday. March and November begin with the same week day as February of an ordinary year, as the interval contains an exact number of weeks and so these months are put down with February. [In a leap year, the 1st of February will begin with an earlier week day and so M in the second column of table III comes against week days beginning with Tuesday and month February (leap) is fixed in the second column. If January 1st is Sunday, 1st April is Saturday as there are (31+28+31=90) 12 weeks and six days between them and so April is put in column (5) having M in the row beginning with Saturday, and the cycle M S E H N O Y is completed for the 5th column of table III. Between January of a leap year and April or July of the same year there are 13 or

26 complete weeks and so these months are also put down in the same column. If April 1st is Saturday, first of May is Monday, and so against the row of week days beginning with Monday and the month May should be put down M and the cycle of letters completed for column 6. If May 1st is Monday, June 1st is Thursday and M in column 7 of table III should come against June and the row of week days beginning with Thursday. The rest of the letters in that column can be filled up following the cyclic order. If June 1st is Thursday, August 1st is Tuesday, and so August is put in the 2nd column having M against the row of week days beginning with Tuesday. If August 1st is Tuesday, September 1st is Friday, and so September is put in the 4th column already having M in the row of week day beginning with Friday. September and December differ by an exact number of weeks; so December is put in the same column as September.

This calendar which has been fitted to agree with that of 1899 should be true for all years.

To facilitate direct reading of the calendar of a month, keep only the particular row of week days in view; this is best done by a sliding board provided, with a slot which can be brought into position showing only the row of week days pertaining to the required month.

Julian Calender (*Old Style*)

The Gregorian Calendar came into operation in England and her colonies on the 14th September 1752, and accordingly the century years in table I begin from 18. The Julian Calendar which was in operation before that time reckoned all century years as leap years (irrespective of the divisibility of the century number by 4) and ended on the 2nd Sept. 1752, the days intervening between these dates being suppressed.

To find the day corresponding to a date of the Julian Calendar, from the above design, it is necessary to insert the century numbers 0 to 17 in Table I as follows, in 7 columns under the successive columns of Table III. Under col. I insert 2, 9, 16; under col. (ii) insert 4, 11; under col. (iii) insert 5, 12; under col. (iv) insert 0, 7, 14; under col. (v) insert 1, 8, 15. Similarly under the next two columns we insert respectively 3, 10, 17 and 6, 13 respectively. It will be seen that the century numbers have to be arranged in 7 columns while only 4 columns suffice for the Gregorian Calendar.

A PERPETUAL CIVIL CALENDAR

III

II

																		Jan. Oct.	Aug. Feb. (Leap)	Feb. Mar. Nov.	Sep. Dec.	Apr. Jul. Jan. (Leap)	May.	Jun.
93		82	76	71	65		54	48	43	37		26	20	15	09		99	M	O	N	E	S	Y	H
94	88	83	77		66	60	55	49		38	32	27	21		10	04		S	Y	O	H	E	M	N
95	89		78	72	67	61		50	44	39	33		22	16	11	05		E	M	Y	N	H	S	O
	90	84	79	73		62	56	51	45		34	28	23	17		06	00	H	S	M	O	N	E	Y
96	91	85		74	68	63	57		46	40	35	29		18	12	07	01	N	E	S	Y	O	H	M
97		86	80	75	69		58	52	47	41		30	24	19	13		02	O	H	E	M	Y	N	S
98	92	87	81		70	64	59	53		42	36	31	25		14	08	03	Y	N	H	S	M	O	E

I

2			0	1	3	
9	4	5	7	8	10	6
16	11	12	14	15	17	13
18	19	20	21			
22	23	24	25			
26	27	28	29			
30	31	32	33			
34	35	36	37			
38	39	40	41			
42	43	44	45			
46	47	48	49			

Directions for fixing the Calendar of any month of a year :

1. Find the letter of Table III corresponding to the century number of the required year from I and the remaining years from II.
2. Fix this letter against the month required in III itself, and move the Date-slide so as to bring the slot against this letter.

Sun	Mon	Tue	Wed	Thu	Fri	Sat
Mon	Tue	Wed	Thu	Fri	Sat	Sun
Tue	Wed	Thu	Fri	Sat	Sun	Mon
Wed	Thu	Fri	Sat	Sun	Mon	Tue
Thu	Fri	Sat	Sun	Mon	Tue	Wed
Fri	Sat	Sun	Mon	Tue	Wed	Thu
Sat	Sun	Mon	Tue	Wed	Thu	Fri

DATE SLIDE

SLOT						
1	**2**	**3**	**4**	**5**	**6**	**7**
8	**9**	**10**	**11**	**12**	**13**	**14**
15	**16**	**17**	**18**	**19**	**20**	**21**
22	**23**	**24**	**25**	**26**	**27**	**28**
29	**30**	**31**				

PERPETUAL CALENDAR

DESIGNED BY MR. K. SUBRAMANIA IYER B.A., LT., RETIRED HEADMASTER, TRIVANDRUM.

MONTHS									Jan. Oct.	May.	Aug. Feb. (leap)	Feb. Mar. Nov.	June.	Sep. Dec.	April. July. Jan. (leap)
YEAR NUMBER									20	19		18		21	
00		23	34	45	56		79	90	Su	M	T	W	Th	F	Sat
01	12		35	46	57	68		91	M	T	W	Th	F	Sat	Su
02	13	24		47	58	69	80		T	W	Th	F	Sat	Su	M
03	14	25	36		59	70	81	92	W	Th	F	Sat	Su	M	T
	15	26	37	48		71	82	93	Th	F	Sat	Su	M	T	W
04		27	38	49	60		83	94	F	Sat	Su	M	T	W	Th
05	16		39	50	61	72		95	Sat	Su	M	T	W	Th	F
06	17	28		51	62	73	84		Su	M	T	W	Th	F	Sat
07	18	29	40		63	74	85	96	M	T	W	Th	F	Sat	Su
	19	30	41	52		75	86	97	T	W	Th	F	Sat	Su	M
08		31	42	53	64		87	98	W	Th	F	Sat	Su	M	T
09	20		43	54	65	76		99	Th	F	Sat	Su	M	T	W
10	21	32		55	66	77	88		F	Sat	Su	M	T	W	Th
11	22	33	44		67	78	89		Sat	Su	M	T	W	Th	F

Directions:—1. The "Year line" starts with the day common to the century number (19 in 1943) given in the second row on the right and the year number (43 in 1943)

2. The week-days in this line give the 1st days of the months entered above in row 1 on the right.

3. Every 5th century repeats the first.

Example 1. Suppose the Calendar for 1887 is desired. The common day for 18 (Century Number) and 87 is Saturday. Therefore the "Year-line" is Sat, Su, M, T, W, Th, F.

Sat — Jan. 1 Oct 1 T — Feb. 1 March 1 and Nov. 1
Su — May. 1 W — June 1
M — Aug. 1 Th — Sept. 1, Dec. 1.
F — April. 1 & July 1.

Example 2. Calendar for leap year 1888. Common day for 18 & 88 = Monday. Year line—M, T, W, Th, F, Sat, Su. Jan. 1 on Sunday (not Monday) Feb. 1 on Wednesday (not Thursday) Other days as above.

THE PERPETUAL CALENDAR PROBLEM AND ITS SOLUTIONS

A Note By the Editor

As there are 7 week days which recur in the same cyclic order, the problem of finding the day corresponding to any date is that of finding the number of intervening days from the week day of a known date, reduced modulo 7. Complications are introduced by the different number of days in different months, and by leap years.

The usual calendars preserve the order of the week days beginning with Sunday, and so require 12 sheets for the 12 months. In Dr. Subramania Iyer's Perpetual Calendar, one date sheet serves for all the 12 months, the week day scheme being varied according to the month, the whole machinery of the calendar being designed to find the week day of the first of the month. It has thus all the advantages of the usual wall calendars except that we do not know when the month ends. The other calendar sent independently by Mr. K. Subramania Iyer of Trivandrum is not essentially different from the first, and is more compact but gives only the first day of each month. The former applies also to the Julian Era, the period before Pope Greogry introduced the reform of regarding century years like 1600, 1700, 1800 etc. as leap years only when the first two digits (century number) is divisible by 4, thus losing 3 days in 4 centuries and reducing the average civil year to one with an error of only 1 day in about 3500 tropical years.

Since the interval between two dates is made up of a number of days, months, years, and centuries, the formula for the week-day corresponding to any date can be expressed in the form

$$W=D+M+C+Y$$

where W is the week day number counting from Sunday as 1, Monday as 2 etc. and D, M, C, and Y are the contributions made by the date, month, century and year numbers. Thus D may be taken to be the actual date reduced mod. 7. One such formula given by that Prince of Mathematicians Karl Friedrich. Gauss is used in the following Perpetual Calendar Table which applies both to Julian and Gregorian Calendars :

Week days	Contribution	Date	Month	Century Greg.	Century Julian	Year
Saturday	0	$7n+0$	October	$4n+0$	$7n+5$	$28n+0$, 6, 17, 23
Sunday	1	1	Jan. May	3	4	1, 7, 12, 18
Monday	2	2	August	—	3	2, 13, 19, 24
Tuesday*	3	3	Mar. Nov.	2	2	3, 8, 14, 25
Wednesday	4	4	Feb. June	—	1	9, 15, 20, 26
Thursday	5	5	Sept. Dec.	1	0	4, 10, 21, 27
Friday	6	6	April, July	—	6	5, 11, 16, 22

In this table the value of D, M, C and Y are those in the column marked "contribution" opposite to the corresponding date, month, century or year. Thus to find the day corresponding to the 15th. Oct. 1582 (the date of the

introduction of the Gregorian Calendar), we take $D=15=1$ (mod 7), $M=0$ for October, $C=1$ since 15 is of the form $4n+3$, and $Y=4$ since 82 is of the form $28n+26$. Therefore $W=6$, and so it was a Friday.

As another example take the 12th. Oct. 1492 (Discovery of America). Here we use the Julian calendar and get $D=5$, $M=0$. $C=5$ since 14 is of the form $7n$ and $Y=3$ since 92 is of the form $28n+8$. Therefore $W=13=6$ so that it is also a Friday.

It should not be difficult to devise a calculating machine for giving the day of the week corresponding to a given date. In place of the usual number wheels, it will have several 7 toothed wheels, the last one carrying on its periphery the 7 days of the week, only one being visible at a time through a window. There will be 4 levers each with 7 possible positions, and if the dates, months etc. are arranged as in the above table (the months being therefore not in serial order, nor the years), it will be only necessary to ensure that each forward step of any of the levers is transmitted to the final wheel which thus gives their sum mod. 7. The century lever for Greorian Calendar will have however only the positions noted in the table, the change from one position to the next moving the last wheel through 2 teeth except the step from the initial to the next position which will move it one division.

Another formula for W which looks simple is the following

$$W=d+2m+\left[\frac{3(m+1)}{5}\right]+y+\left[\frac{y}{4}\right]-c-g.$$

Here $[x]$ stands for the greatest integer not exceeding x, d is the date reduced mod. 7, m is the number of the month in the year (January and February being however reckoned as the 13th and 14th months of the preceeding year) c the century and y the year number, and $g=0$ for the Julian Calendar while for the Gregorian

$$g=c-\left[\frac{c}{4}\right]-2$$

Thus for the 11th Nov. 1918, the date of the Armistice,

$$W=4+1+[36/5]+18+[18/4]-19-g$$

where $g=13$ Therefore $W=2$ mod. 7. It was a Monday.

Nomograms have also been constructed for obtaining the day when the date is given. Thus in the accompanying one, we have 4 coloumns of dots to get the date we first align the date (left of first column) and the month (third column), and get its intersection with column 2. Join this point to the century number (fourth column, read left or right according as it is Gregorian or Julian) and get its intersection with column 3. Lastly join this point with the point corresponding to the year (right of col. 1) and get the intersection with col. 2 to get the week day. The 4 vertical lines are equidistant, and the points are spaced uniformly on each, the distances on the first and third being half that on the fourth line, those on the second line ¼ of those on the last.

I am indebted for much of this information to Kraitchik's excellent work "La Mathematique des jeux ou Recreations Mathematiques", of which I learn that an English translation will soon be published in America. The formula for W has been however, somewhat simplified.

Since this is a continental publication, the Gregorian Calendar holds from the 15th October 1582, whereas the change over was postponed in Britain, her colonies and North America till 1752.

DATE	YEAR	DAY	MONTH	CENTURY NUMBER Greg.	CENTURY NUMBER Julian
		•			
	• $28n+22$	•	•		
		W •			
	• ,, 27	• T	•		• 13
		M •			
	• ,, 26	• Su	•		
		S •			
6, 13, 20, 27	• ,, 25	• F	Apr • July	21 •	7
		Th •			
	• ,, 24	• W	•		
		T •			
	• ,, 18	• M	•		• 8
		Su •			
	• ,, 23	• S	•		
		F •			
5, 12, 19, 26	• ,, 16	• Th	Sep • Dec	22 •	9
		W •			
	• ,, 21	• T	•		
		M •			
	• ,, 20	• Su	•		• 10
		S •			
	• ,, 14	• F	•		
		Th •			
4, 11, 18, 25	• ,, 19	• W	Feb • June	15 •	11
		T •			
	• ,, 12	• M	•		
		Su •			
	• ,, 17	• S	•	20 •	12
		F •			
	• ,, 11	• Th	•		
		W •			
3, 10, 17, 24, 31	• ,, 10	• T	Mar • November		• 6
		M •			
	• ,, 15	• Su	•		
		S •			
	• ,, 8	• F	•	17 •	14
		Th •			
	• ,, 13	• W	•		
		T •			
2, 9, 16, 23, 30	• ,, 7	• M	• August		• 1, 15
		Su •			
	• ,, 6	• S	•		
		F •			
	• ,, 5	• Th	•	18 •	2
		W •			
	• ,, 4	• T	•		
		M •			
1, 8, 15, 22, 29	• ,, 9	• Su	Jan • May		• 3
		S •			
	• ,, 3	• F	•		
		Th •			
	• ,, 2	• W	•	19 •	4
		T •			
	• ,, 1	• M	•		
		Su •			
7, 14, 21, 28	• 2, $8n$	• S	• October	16 •	5

SOME INEQUALITIES INVOLVING THE Γ-AND ζ-FUNCTIONS

BY

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In this note certain inequalities involving the Γ and ζ Functions are obtained by elementary methods.

1. We have if $p,\ q > 0$,

$$B(p, q) = \int_0^1 x^{p-1}\ (1-x)^{q-1}\ dx.$$

Let $F(x) = x^r\ (1-x)^s$, r,s being either both positive or both negative.

Then, if r,s are positive (negative),

$$\max\ (\min)\ F(x) = \left(\frac{r}{r+s}\right)^r \left(\frac{s}{r+s}\right)^s$$

Hence, if $0<p>r,\ 0<q>s$,

$$B(p, q) \lessgtr \left(\frac{r}{r+s}\right)^r \left(\frac{s}{r+s}\right)^s B(p-r,\ q-s)$$

according as $r,\ s \gtrless 0$.

If $p>r\geqslant 0,\ q>s\geqslant 0$, writing

$$\gamma\ (p, r) = \Gamma(p)/r^r\,\Gamma p - r)$$

the above result may be written

$$\gamma\ (p,r)\,\gamma\,(q, s) < \gamma\,(p+q,\ r+s)$$

Corollary: Putting $s=0$, we get, if $p> r\geqslant 0$, $q>0$, $\gamma(p,\ r)<\gamma(p+q,\ r)$, that is to say, $\gamma\ (x,r)$ is monotonic increasing as x increases from r (>0) to infinity. This implies that $\Gamma(x)/\Gamma(x-r)$ is monotonic increasing as x increases from r (>0) to infinity.

From this we get,

$\Gamma'(x)/\Gamma(x) > \Gamma'(x-r)/\Gamma(x-r)$ if $x>r>0$, which is equivalent to saying that $\psi\ (x) = \Gamma'\ (x)/\Gamma\ (x)$ is monotonic increasing as x increases from 0 to ∞.

2. We next consider the integral

$$\Gamma\ (p) = \int_0^\infty e^{-x}\ x^{p-1}\ dx \quad , \quad p>0.$$

A simple transformation gives

$$\Gamma(p) = \int_0^{1/n} e^{-x/(1-nx)}\, x^{p-1} (1-nx)^{-p-1}\, dx.$$

Noting that the max. of $e^{-x/(1-nx)} (1-nx)^{-t}$ is $e^{\frac{1}{n}-t} (nt)^t$, we get

$$\Gamma(p) < e^{\frac{1}{n}-t} (nt)^t \int_0^{\frac{1}{n}} x^{p-1} (1-nx)^{t-p-1}\, dx \text{ , if } t > p > 0$$

$$= e^{\frac{1}{n}-t} (nt)^t\, n^{-p}\, \mathrm{B}(p, t-p)$$

This gives $\Gamma(t) < e^{\frac{1}{n}-t}\, n^{t-p}\, t^t\, \Gamma(t-p)$ if $t>p>0$.

But min. $e^{\frac{1}{n}}\, n^{t-p}$ is $e^{t-p} (t-p)^{p-t}$, so that

$$\Gamma(t) < \left(\frac{t}{e}\right)^t e^{t-p}\, \Gamma(t-p)/(t-p)^{t-p} \text{ , } t>p>0,$$

which may be written as

$$\left(\frac{e}{t}\right)^t \Gamma(t) < \left(\frac{e}{t-p}\right)^{t-p} \Gamma(t-p) \text{ , } t > p > 0.$$

This implies that

$\Gamma(x) \left(\frac{e}{x}\right)^x$ is monotonic decreasing as x increases from 0 to ∞. From this we easily get

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} < \log x, \text{ if } x > 0.$$

3. If $\zeta(s, a) = \sum_{n=1}^{\infty} \frac{1}{(n+a)^s}$, $s > 1$, $a > -1$,

we have

$$\Gamma(s)\, \zeta(s, a) = \int_0^1 \left(\log \frac{1}{x}\right)^{s-1} \frac{x^a\, dx}{1-x}$$

Let $\varphi(x) = \left(\log \frac{1}{x}\right)^p x^q$, p and q having the same sign.

Then, according as $p, q > 0$ (or < 0) we get

max. (or min.) $\varphi(x) = (p/eq)^p$.

Accordingly we get, if $s > \max(p+1, 1)$, and $a > \max(q+1, 1)$, then

$$\Gamma(s)\, \zeta(s, a) \lesseqgtr \int_0^1 \left(\frac{p}{eq}\right)^p \left(\log\frac{1}{x}\right)^{s-p-1} \frac{x^{a-q}}{1-x}\, dx$$

$$= \left(\frac{p}{eq}\right)^p \Gamma(s-p)\, \zeta(s-p, a-q)$$

the upper sign holding when $p, q > 0$ and the lower when $p, q < 0$.

Corollary. Taking $a=q=1$ we get , if $s > p+1 > 1$,

$$\Gamma(s)\{\zeta(s)-1\} < (p/e)^p\, \Gamma(s-p)\, \zeta(s-p),$$

where
$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s .$$

A METHOD OF EXTRACTING THE nth ROOT OF A POSITIVE NUMBER

BY

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1. I give below a method of extracting the nth root of a positive number expressed in the usual decimal scale. It is an extension of the ordinary process of extracting the square root. The process is explained in section **2**. The next three sections are devoted to specific examples. In the last section, a proof of the validity of the process is given.

2. Let a be the given number expressed in the decimal system and let its nth root be required. Mark off groups of n digits to the left starting from the units digit, and to the right starting from the first decimal place adding zeros or groups of zeros when necessary. Let R_0 denote the integer formed by the last group or (part of a group left over) after marking off groups of n digits to the left. Then the successive digits $u_0, u_1, u_2, \ldots$ of the nth root can be obtained as follows :—

(i) Let u_0 be the greatest integer such that $u_0^n \leqslant R_0$. Then u_0 is the first digit counting from the left. Let $D_0 = u_0^{n-1}$. Divide R_0 by D_0 and to the right of the remainder obtained bring down the next group of n digits (as in the process of extracting square roots). Let the number thus obtained be R_1.

(ii) Let $Q_1 = 10u_0$. Let u_1 be the greatest integer such that $(u_1 + Q_1)^n - Q_1^n \leqslant R_1$. Then u_1 is the second digit counting from the left. Let $D_1 = \frac{1}{u_1}\left\{ (u_1 + Q_1)^n - Q_1^n \right\}$. Divide R_1 by Q_1, and to the right of the remainder bring down the next group of n digits. Let R_2 denote the number thus obtained.

(iii) Let $Q_2 = 10^2u_0 + 10u_1 = 10 \times$ (the number formed by the digits u_0 and u_1 already found) Let u_2 be the greatest integer such that $(u_2 + Q_2)^n - Q_2^n \leqslant R_2$. Then u_2 is the third digit in the nth root counting from the left. Divide R_2 by $D_2 = \frac{1}{u_2}\left\{ (u_2 + Q_2)^n - Q_2^n \right\}$ and bring down the next group of n digits and let R_3 denote the number thus formed.

(iv) Let $Q_3 = 10^3u_0 + 10^2u_1 + 10u_2 = 10 \times$ (the number formed of the three digits u_0, u_1, u_2, already found). We can now repeat the process with R_3 and Q_3 and obtain u_4 and so on. We can ignore the decimal point in the given number a and deal with integers alone in the above process, and finally put the decimal point in the final answer at the proper place as in the extraction of square root. If the process comes to a stop after a finite number of repetitions of the above process, the given number is the nth power of a rational number. This will be the case when one of R_1, R_2, . . is identically zero. Othewise the process continues indefinitely and the nth root can be obtained to any number of places of decimals.*

(v) The quantities R_0, R_1, R_2, . . may be called the successive reminders, $Q_0 = 0$, $\frac{Q_1}{10}$, $\frac{Q_2}{10}$, $\frac{Q_3}{10}$, . . . the successive quotients and D_0, D_1, D_2, . . the successive divisors, in the above process.

3. To find the cube root of 9,420,668,928. Here $R_0 = 9$.

(i) u_0 is the greatest integer such that $u_0^3 \leqslant 9$. Hence $u_0 = 2$, $D_0 = 4$ and $R_1 = 1420$.

(ii) u_1 is the greatest integer such that $(u_1 + 20)^3 - 20^3 \leqslant 1420$ This gives $u_1\ (u_1^2 + 60u_1 + 1200) \leqslant 1420$. That is, $u_1 = 1$, $D_1 = 1261$ so that $R_2 = 159668$.

(iii) u_2 is the greatest integer so that $(u_2 + 210)^3 - (210)^3 \leqslant 159668$. That is, $u_2(u_2^2 + 630u + 132300) \leqslant 159668$. This gives $u_2 = 1$, $D_2 = 132931$ so that $R_3 = 26737928$.

* It may be noted that $1 \leqslant u_0 \leqslant 9$, $0 \leqslant u_k \leqslant 9$, and $k = 1, 2, 3, \ldots$ as may be easily seen from the method of forming R_0, R_1, R_2, . . and Q_0, Q_1, Q_2. . . .

(iv) u_3 is the greatest integer such that $(u_3+2110)^3-(2110)^3 \leqslant 26737928$. This gives $u_3=2$, $D_3=13368964$. Also $2D_3=R_3$ so that $R_4=0$.

So the cube root is 2112.

4. To find the cube root of $a=2$. Here $R_0=2$

(i) Here $u_0=1$, $D_0=1$, $R_1=1000$. The decimal point in the cube root to be placed after the digit $u_0=1$.

(ii) $u_1=2$, $D_1=364$, $R_2=272000$.

(iii) $u_2=5$, $D_2=45025$. $R_3=46875000$.

(iv) $u_3=9$, $D_3=4721331$. $R_4=4383021000$.

(v) $u_4=9$, $D_4=475864311$, $R_5=100242201000$.

(vi) $u_5=2$, etc.

Hence the cube root of 2 up to four decimal places is $1\cdot2599$.

5. To find the fifth root of $a=4084101$. Here $R_0=40$.

(i) $u_0=2$, $D_0=16$, $R_1=884101$.

(ii) u_1 is the greatest integer such that $(u_1+20)^5-20^5 \leqslant 884101$.

This gives

$$u_1\ (u_1^4+5\ u_1^3\cdot 20+10\ u_1^2\cdot 20^2+10\ u_1\cdot 20^3+5\cdot 20^4) \leqslant 884101.$$

By trial we find $u_1=1$ and $D_1=884101$ so that $R_2=0$. Hence 21 is the fifth root of the given number.

Proof of the method

6. We shall give the proof for $u=3$. The proof is, however, perfectly general and applies to any n. Let

$$10^p\ n_0+10^{p-1}\ u_1+\ \cdot\ \cdot\ +\ u_p+\frac{u_{-1}}{10}+\frac{u_{-2}}{10^2}+\ \cdot\ \cdot$$

be the cube root of a number a. Then a will be of the form

$$10^{3p+2}\ a_0+10^{3p+1}\ a_1+{}^{3p+2}\ 10^{3p}\ a_2+\ \cdot\ \cdot\ \cdot\ \cdot\ +\frac{a_{-1}}{10}+\frac{a_{-2}}{10^2}+\ \cdot\ \cdot\ \cdot$$

where a_0, a_1, a_2 cannot all be simultaneously zero.

Then we have the identity

$$(10^p\ u_0+\ \cdot\ \cdot\ )^3=10^{3p+2}\ a_0+10^{3p+1}\ a_1+10^{3p}\ a_2+\ \cdot\ \cdot\ \cdot \qquad \ldots\ (1)$$

Dividing by $(10^p)^3=10^{3p}$ we get

$$\left(u_0+\frac{u_1}{10}+\ \cdot\ \cdot\right)^3=10^2\ a_0+10\ a_1+a_2+\frac{a_3}{10}+\ \cdot\ \cdot\ \cdot$$

The gives

$$u_0^3 \leqslant 10^2\ a_0 + 10\ a_1 + a_2 < (u_0+1)^3 \qquad \ldots \quad (2)$$

which proves the statement in (i) of §2 since $R_0 = 10^2\ a_0 + 10\ a_1 + a^2$.

Next divide (1) by $(10^{p-1})^3 = 10^{3p-3}$. Then we get

$$\left(10u_0 + u_1 + \frac{u_2}{10} + \cdot\cdot\right)^3 = 10^5\ a_0\ + \cdot\cdot\cdot + a_5 + \frac{a_6}{10} + \cdot\cdot\cdot$$

which gives

$$(10\ u_0 + u_1)^3 \leqslant 10^5\ a_0 + \ \cdot\cdot\ + a_5 < (10\ u_0 + u_1 + 1)^3 \qquad \ldots \quad (2)$$

That is u_1 is the greatest integer so that,

$$(u_1 + 10\ u_0)^3 \leqslant 10^3\ (10^2\ a_0 + 10\ a_1 + a_2) + 10^2\ a_3 + 10\ a_4 + a_5.$$

Hence since $Q_1 = 10\ u_0$, we get that u_1 is the greatest integer such that

$$(u_1 + Q_1)^3 - Q_1^3 \leqslant 10^3\ (10^2\ a_0 + 10\ a_1 + a_2) + 10^2\ a_3 + 10\ a_4 + a_5 - Q_1^3.$$

But $R_0 = 10^2 a_0 + 10 a_1 + a_0$ and $R_0 - D_0 u_0 = R_0 - u_0^3$. Combining these we get that u_1 is the greatest integer such that

$$(u_1 + Q_1)^3 - Q_1^3 \leqslant 10^3\ (R_0 - D_0\ u_1) + 10^2\ a_3 + 10\ a_4 + a_5$$

which is exactly the statement in (ii) of §2. Similarly dividing by $(10^{p-2})^3$, $(10^{p-3})^3$, . . . successively the equality (1) we can prove the statements in (iii), (iv), . . . of §2. As already pointed out the decimal point may be ignored in the calculation of u_0, u_1, u_2, . . . successively.

ON SOME PROBLEMS IN THE THEORY OF EQUATIONS

BY

AMRITSAGAR PURI, *Kapurthala.*

1. The object of this paper is to discuss a general method of determining a certain number of roots of an equation, given their sum. It leads naturally to the solution of the cubic and biquadratic and, in some cases, in finding certain roots of equations of higher order. The method is not generally applicable to equations of degree higher than 4 in view of their insolubility by radicals.

2. Suppose the sum of two roots of an equation is given. Let α and β be the roots whose sum $\alpha + \beta = 2\lambda$ is given. Replacing x by $x + iy$, expanding each power of $x + iy$ and collecting together the terms containing i and the remaining terms we can write*

$$f(x+iy) = g(x,\ y) + i\ h(x,\ y).$$

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* The coefficients of $f(x)$ may be real or complex; in the latter case they are *not split up* into real and imaginary parts in getting the equation for $f(x+iy)$ and $f(x-iy)$.

Then we get $f(x-iy)=g(x, y)-ih(x, y)$.

Putting $u=y^2$ we see that the equations

$$g_1(x, u) = g(x, y) = 0; \; h_1(x, u) = \frac{h(x, y)}{y} = 0$$

are satisfied by $x=\frac{1}{2}(\alpha+\beta)$ and $u=\frac{1}{4}(\alpha-\beta)^2$. This is true even when $\alpha=\beta$ as may be seen by evaluating $g(x, y)$ and $h(x, y)$ in terms of $f(x+iy)$ and differentiating. Now, since $\alpha+\beta=2\lambda$ is given we see that u must satisfy simultaneously the equations

$$g_1(\lambda, u)=0, \; h_1(\lambda, u)=0$$

If $u_1, u_2, \ldots, u_p$ be the common roots of these equations, then $\lambda \pm i\sqrt{u_k}$, $k=1, 2, \ldots, p$ are the pairs, the sum of each pair being λ

2·1. In this connection it is easy to prove that (*i*) if an equation $f(x)=0$ of degree $2n$ has n pairs of roots, the sum of each pair being 2λ, then $f'(\lambda)=0$; and (*ii*) if $f(x)=0$ of degree $2n+1$ has n such pairs, then λ is a root of $[(2n+1)x+a]f'(x)-f(x)=0$ where a is the coefficient of x^{2n} in $f(x)$.

2·2. The method in §2 leads naturally to the usual solution of the quadratic. Let us consider the application of this to the quintic

$$x^5+px+q=0. \qquad \ldots (1)$$

Here calculation gives

$$\left.\begin{aligned} g_1(x, u) &= x^5-10x^3u+5xu^2+px+q \\ h_1(x, u) &= 5x^4-10x^2u+u^2+p. \end{aligned}\right\} \qquad \ldots (2)$$

Hence if the sum of two of the roots be known, say 2λ, the value of u can be easily found from (2) to be

$$u=\frac{24\lambda^5+4p\lambda-q}{40\lambda^3}.$$

Since every quintic can be reduced to the form (1), it follows that a quintic can be solved by radicals if and only if the equation of degree ten whose roots are the sums, two by two, of the roots of (1) has a root expressible by radicals. Hence if a quintic is not soluble by radicals, the sum of no two of its roots can be expressed by radicals.

3. We next consider the case when the sum of three roots of an equation is known. We require some preliminary results. Let

$$f(x)=x^n+p_1x^{n-1}+\cdots+p_n=0$$

be the equation. Replacing x by $x+\omega y+\omega^2 z$ in $f(x)$, expanding each

power of $x+\omega y+\omega^2 z$ and collecting the coefficients of ω, ω^2 and the independent term we can write

$$f(x+\omega y+\omega^2 z)=f_1(x, y, z)+\omega f_2(x, y, z)+\omega^2 f_3(x, y, z)$$

where ω is a primitive cube root of unity. Then we get

$$f(x+\omega^2 y+\omega z)=f_1(x, y, z)+\omega^2 f_2(x, y, z)+\omega f_3(x, y, z)$$
$$f(x+y+z)=f_1(x, y, z)+f_2(x, y, z)+f_3(x, y, z).$$

From these we get, conversely,

$$\left.\begin{aligned} f_1(x, y, z)&=\tfrac{1}{3}\,[f(x+y+z)+f(x+\omega y+\omega^2 z)+f(x+\omega^2 y+\omega z)] \\ f_2(x, y, z)&=\tfrac{1}{3}\,[f(x+y+z)+\omega^2 f(x+\omega y+\omega^2 z)+\omega f(x+\omega^2 y+\omega z)] \\ f_3(x, y, z)&=\tfrac{1}{3}\,[f(x+y+z)+\omega f(x+\omega y+\omega^2 z)+\omega^2 f(x+\omega^2 y+\omega z)] \end{aligned}\right\} \quad \ldots \quad (3)$$

From these equations we see that if α, β, γ are any three roots of $f(x)=0$ and a, b, c are defined by

$$a=\tfrac{1}{3}(\alpha+\beta+\gamma);\ b=\tfrac{1}{3}(\alpha+\omega\beta+\omega^2\gamma);\ c=\tfrac{1}{3}(\alpha+\omega^2\beta+\omega\gamma); \quad \ldots \quad (4)$$

then a, b, c satisfy simultaneously the three equations.

$$f_1(x, y, z)=f_2(x, y, z)=f_3(x, y, z)=0 \quad \ldots \quad (5)$$

The equations (4) give

$$\alpha=a+b+c,\ \beta=a+\omega^2 b+\omega c,\ \gamma=a+\omega b+\omega^2 c. \quad \ldots \quad (6)$$

It is easily verified that the necessary and sufficient condition that two out of α, β, γ should be equal to each other is that $b^3-c^3=0$. Now the expression f_1 is symmetric in y and z but f_2 and f_3 interchange when y and z are interchanged. Our next object is to get functions which are symmetric in y and z and at the same time vanish for $x=a$, $y=b$ and $z=c$. Firstly, using the equations (3) we see that zf_2-yf_3 has y^3-z^3 as a factor and if

$$zf_2-yf_3=(y^3-z^3)\,f_4(x, y, z), \quad \ldots \quad (7)$$

then f_4 is symmetric in y and z. Also $f_4=0$ is satisfied by $x=a$, $y=b$, $z=c$. This is seen from (7) when α, β, γ are all distinct since then $b^3-c^3\neq 0$. If α, β, γ are not distinct we use the formulae (3) to show $\frac{\partial}{\partial y}(zf_2-yf_3)=0$ when $y^3-z^3=b^3-c^3=0$. Hence in this case also $f_4=0$ when $x=a$, $y=b$, $z=c$.

Next, it is easily verified that

$$\frac{1}{y}\left[f_2+z^2 f_4\right]=\frac{1}{z}\left[f_3+y^2 f_4\right]$$

and if the common function be denoted by $f_5(x, y, z)$, then $f_5=0$ is also satisfied by $x=a$, $y=b$, $z=c$. Here also the case $b=0$ or $c=0$

can be dealt with as above by differentiation. Thus we get that $x=a$, $y=b$, $z=c$, satisfy simultaneously the three equations

$$f_1=0,\ f_4=0,\ f_5=0. \qquad \dots \quad (8)$$

where all the functions are symmetric in y and z.

3·1. We can now proceed to discuss the method of finding α, β, γ given that $\alpha+\beta+\gamma=3\lambda$, say. By (8), we have the three equations

$$f_1\ (\lambda,\ y,\ z)=0,\ f_4\ (\lambda,\ y,\ z)=0,\ f_5\ (\lambda,\ y,\ z)=0. \qquad \dots \quad (9)$$

If we eliminate z between f_1 and f_4 and between f_1 and f_5 we shall get two equations

$$f_6\ (y)=0,\ f_7\ (y)=0. \qquad \dots \quad (10)$$

In view of the fact that f_1, f_4, f_5 are symmetric in y and z, the equation (10) will be satisfied by $y=b$ and $y=c$. Hence b and c are the common roots of the two equations (10). Having found b and c and noting that $a=\lambda$ we can find α, β, γ from the equations (6). It must be noted that a pair of common roots of (10) must be the values of b and c given that there are three roots of $f(x)=0$ whose sum is 3λ. To determine exactly which pair it is, we can actually substitute in $f(x)=0$ and ascertain whether $a+b+c$ is a root. If so, the other two roots are $a+\omega^2\ b+\omega c$ and $a+\omega b+\omega^2\ c$.

3·2. Now consider the expression $p+\omega q+\omega^2\ r$. If p, q, r are permuted we get 6 values which fall into 2 pairs of 3 each as follows :—

$p+\omega q+\omega^2 r$, $\omega\ (p+\omega q+\omega^2 r)$, $\omega^2\ (p+\omega q+\omega^2 r)$ and $p+\omega^2 q+\omega r$, $\omega\ (p+\omega^2 q+\omega r)$, $\omega^2\ (p+\omega^2 q+\omega r)$ From this we see that the six values are the roots of the quadratic in t^3, namely

$$[\ t^3-(p+\omega q+\omega^2 r)^3\ ]\ [\ t^3-(p+\omega^2 q+\omega r)^3\ ]=0$$

which is an equation of the form

$$At^6+Bt^3+C=0.$$

using this fact we easily see that the common factors of (10) must be factorisable into factors of the form

$$Ay^6+By^3+C=0 \qquad \cdot \quad \cdot \quad \cdot \quad (11)$$

and b^3 and c^3 are the roots of this quadratic in y^3. Hence the three roots of $f(x)=0$ will be found among the nine values of the expression $\lambda+p^{1/3}+q^{1/3}$ where p and q are the roots of $At^2+Bt+C=0$ where A, B, C are given by (11). Either by trial or by other supplementary relations in special cases we can ascertain which triad actually are the roots of $f(x)=0$. It may also be noted that if a particular

combination, say, $\lambda+\sqrt[3]{p}+\sqrt[3]{q}$ satisfies $f(x)=0$, then the remaining two are $\lambda+\omega\sqrt[3]{p}+\omega^2\sqrt[3]{q}$ and $\lambda+\omega^2\sqrt[3]{p}+\omega\sqrt[3]{q}$.

3·3. Since the sum of the three roots of a cubic is always known the above method can be applied and leads to the usual Cardan's solution. Let us consider now the biquadratic

$$x^4+px+q=0$$

for which $\alpha+\beta+\gamma=3\lambda$ is supposed to be known.

Here

$$f_1=x^4+12x^2yz+4x(y^3+z^3)+px+6y^2z^2+q$$
$$f_2=y^4+12y^2xz+4y(x^3+z^3)+py+6x^2z^2$$
$$f_3=z^4+12z^2xy+4z(x^3+y^3)+pz+6x^2y^2$$

From these we get

$$zf_2-yf_3=-(y^3-z^3)(6x^2+3yz)$$

so that $f_4=-(6x^2+3yz)$

Also $f_5=\dfrac{1}{y}[f_2+z^2f_4]=4x^3+y^3+z^3+12xyz+p.$

Thus we see that b and c satisfy the three equations

$$\left.\begin{aligned}\lambda^4+12\lambda^2yz+4\lambda(y^3+z^3)+p\lambda+6y^2z^2+q=0\\ 2\lambda^2+yz=0\\ 4\lambda^3+y^3+z^3+12\lambda yz+p=0.\end{aligned}\right\}\quad\dots\quad(12)$$

These three give

$$yz=-2\lambda^2$$
$$y^3+z^3=20\lambda^3-p$$

That is b^3 and c^3 are the roots of the quadratic

$$t^3-(20\lambda^3-p)t-8\lambda^6=0$$

Also λ satisfies the equation

$$81\lambda^3-4p\lambda+q=0$$

as may be seen by substituting for y^3+z^3 and yz in the first of the equations (12). This can also be derived as the equation whose roots are one third the sum, three by three, of the roots of the original equation.

3·4. Consider the equation

$$x^4+2x+1=0$$

One root is -1. Hence the sum of the remaining roots is 1 so that $\lambda=\frac{1}{3}$. Thus b^3 and c^3 are the roots of the quadratic

$$(27t)^2+34\ (27t)-8=0.$$

The roots are

$$\frac{1}{27}\ (-17+\sqrt{297})\ \text{and}\ \frac{1}{27}\ (-17-\sqrt{297})$$

Hence b is one of the values of $\frac{1}{3}\ (-17+\sqrt{297})^{\frac{1}{3}}$ and c that of $\frac{1}{3}\ (-17-\sqrt{297})^{\frac{1}{3}}$. But $bc=-\frac{2}{9}$ so that b and c may be taken as the real cube roots.

Hence the three roots of $x^4+2x+1=0$ apart from -1 are $\frac{1}{3}(1+\mu+\nu)$, $\frac{1}{3}(1+\omega\mu+\omega^2\nu)$, $\frac{1}{3}(1+\omega^2\mu+\omega\nu)$ where μ and ν are the real cube roots of $-17+\sqrt{297}$ and $-17-\sqrt{297}$ respectively.

4. The above method can be extended to the case when the sum of p roots of an equation is given by using a corresponding primitive pth root of unity. But when $p>4$, and even when $p=4$, the method is cumbersome and it might be applicable only in special cases.

GLEANING

GOD EXISTS

Invited by Catherine the Great to visit her court, Diderot (a French Philosopher 1713—1784) earned his keep by trying to convert the courtiers to atheism. Fed up, Catherine commissioned Euler to muzzle the windy philosopher. This was easy because all mathematics was Chinese to Diderot. De Morgon tells what happened (in his classic *Budget of Paradoxes*, 1872): "Diderot was informed that a learned mathematician was in possession of an algebraical demonstration of the existence of God, and would give it before all the Court, if he desired to hear it. Diderot gladly consented. . . . Euler advanced towards Diderot, and said gravely in a tone of perfect conviction:

"*Sir* $\frac{a+b^n}{n}=x$, *hence God exists; reply!*"

It sounded like sense to Diderot. Humiliated by the unrestrained laughter which greeted his embarrassed silence, the poor man asked Catherine's permission to return at once to France. She graciously gave it,

Not content with this masterpiece, Euler in all seriousness painted his lily with solemn proofs, in deadly earnest, that God exists and that the soul is not a material substance. It is reported that both proofs passed into the treatises in Theology of his day. They are probably the choicest flowers of the mathematically unpractical side of his genius.

From BELL; *Men of Mathematics.*

A NEW SUMMATION PROCESS

BY

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1. I introduce here a new method of *summing* infinite series—called *Bessel Summability*—which I had to discuss in my D. Sc. thesis while studying the problem of expanding an arbitrary function in a series of eigenfunctions. The process does not seem to be deeper than summability by typical means. But still the method is, in my opinion, interesting in itself, in as far as it can be viewed as a generalisation of Lebesgue and Riemann Summabilities,—usually introduced in the theory of Fourier Series.

I give below a few elementary results only, without attempting any systematic study of the subject. A detailed study of a special type of Bessel Summability viz. summability (J_μ, n) was however attempted by K. Chandrasekaran in his M. Sc. Thesis.

1·1. Let $J_\mu(t)$ denote the Bessel function of order μ, $(\mu > -1)$, that is

$$J_\mu(t) = \frac{t^\mu}{2^\mu \Gamma(\mu+1)} \left(1 - \frac{t^2}{2(2\mu+2)} + \cdots \right) \qquad \ldots \quad (1)$$

and let us write

$$\alpha_\mu(t) = \frac{2^\mu \Gamma(\mu+1)}{t^\mu} J_\mu(t) \qquad \ldots \quad (2)$$

Further let $\sum a_n$ denote an infinite series with partial sums $A_n = a_1 + a_2 + \cdots + a_n$, and let λ_n be an increasing sequence of positive numbers diverging to ∞, that is

$$0 < \lambda_1 < \lambda_2 \cdots < \lambda_n \to \infty.$$

DEFINITION I: A series $\sum a_n$ is said to be summable (J_μ, λ) if the series

$$\sum a_n \alpha_\mu(\lambda_n t) = \phi_\mu(t) \qquad \ldots \quad (3)$$

is convergent for $t > 0$, and if

$$\lim_{t \to 0} \phi_\mu(t) = s \qquad \ldots \quad (4)$$

exists, in which case we say s is the *sum** of the series $\sum a_n$

DEFINITION II: A series $\sum a_n$ is said to be summable (J_μ^k, λ), if the series

$$\sum a_n \{\alpha_\mu(\lambda_n t)\}^k = \phi_\mu^k(t) \qquad \ldots \quad (5)$$

* If $\lambda_n = n$ and $\mu = \frac{1}{2}$ we have Lebesgue summability.

is convergent for $t>0$, k being a positive integer and if

$$\lim_{t\to 0} \phi_\mu^k(t) = s \qquad \dots \quad (6)$$

exists.*

N. B.:—In these definitions it is convenient to assume that the series for $\phi_\mu(t)$ is either absolutely convergent for $t>0$, or boundedly convergent for $t>0$. Indeed it is not necessary that the series should be convergent for all $t>0$. It is enough if it so for $0<t\leqslant\delta$, where δ is some positive number.

2. THEOREM 1. If $\sum a_n$ is convergent, than it is summable (J_μ^k, λ) for $k(\mu+\frac{1}{2})>1$. In particular a convergent series is summable (J_μ, λ) for $\mu>\frac{1}{2}$.

Proof:—We have only to observe that the kernal $\alpha_\mu^k(\lambda_n t)$ fulfills Toeplitz condition viz

$$\lim_{t\to 0} \alpha_\mu^k(\lambda_n t) = 1 \qquad \dots \quad (7)$$

$$\sum_{n=1}^{\infty} \left| \alpha_\mu^k(\lambda_n t) - \alpha_\mu^k(\lambda_{n+1} t) \right| < M \qquad \dots \quad (8)$$

where M is a constant independent of t.

The first condition, namely (7), follows from (1) and (2), while (8) will follow if we prove that

$$\int_0^\infty \left| \frac{d}{dx} \alpha_\mu^k(xt) \right| dx < M. \qquad \dots \quad (9)$$

Setting $y=xt$ in (9), we see that it is equivalent to

$$\int_0^\infty \left| \frac{d}{dy} \alpha_\mu^k(y) \right| dy < M. \qquad \dots \quad (10)$$

Now

$$\frac{d}{dy}\left\{ \alpha_\mu(y) \right\}^k = -k \left\{ \alpha_\mu(y) \right\}^{k-1} 2^\mu \Gamma(\mu+1) \frac{J_{\mu+1}(y)}{y^\mu}$$

$$= -k \left\{ 2^\mu \Gamma(\mu+1) \right\}^k \frac{J_\mu^{k-1}(y)\, J_{\mu+1}(y)}{y^{k\mu}} \qquad \dots \quad (11)$$

The expression on the right of (11) tends to zero as $y\to 0$ and is of the order $y^{-k(\mu+\frac{1}{2})}$ as $y\to\infty$.† Therefore the integral (10) will converge provided $k(\mu+\frac{1}{2}) > 1$.

* If $k=2$, $\lambda_n=n$ and $\mu=\frac{1}{2}$ we have Riemann summability.

† As $y\to\infty$, $J_\mu(y)=O\left(y^{-\frac{1}{2}}\right)$ See Watson: *Bessel Functions* § 7·3.

Thus the first part of the theorem is proved. The second part follows from the first when we put $k=1$.

THEOREM 2: If $\sum a_n$ is summable (J_μ, λ), then it is summable (J_ν, λ) to the same sum, for $\nu>\mu$.

Proof:—This theorem is proved with the assumption that the series for $\phi_\mu(t)$ either converges boundedly or absolutely for $t>0$ – that is the convergence is such as to permit term by term integration.

By hypothesis, $\phi_\mu(t)$ is bounded near the origin and hence writing xt for t in (3), multiplying both sides of (3) by the function

$$\frac{2\,\Gamma(\nu+1)}{\Gamma(\nu-\mu)\,\Gamma(\mu+1)}\; t^{2\mu+1}\,(1-t^2)^{\nu-\mu-1}$$

and integrating with respect to t from 0 to 1, we have*

$$\frac{2\,\Gamma(\nu+1)}{\Gamma(\nu-\mu)\,\Gamma(\mu+1)}\int_0^1 \phi_\mu(xt)\,t^{2\mu+1}(1-t^2)^{\nu-\mu-1}\,dt$$

$$=\frac{2\,\Gamma(\nu+1)}{\Gamma(\nu-\mu)\,\Gamma(\mu+1)}\sum a_n\int_0^1 t^{2\mu+1}(1-t^2)^{\nu-\mu-1}\alpha_\mu(\lambda_n xt)$$

$$=\sum a_n\,\alpha_\nu(\lambda_n x)=\phi_\nu(x).$$

That is to say

$$\phi_\nu(x)=\frac{2\,\Gamma(\nu+1)}{\Gamma(\nu-\mu)\,\Gamma(\mu+1)}\,\frac{1}{x^{2\nu}}\int_0^x y^{2\mu+1}(x^2-y^2)^{\nu-\mu-1}\phi_\mu(y)\,dy \quad \ldots (12)$$

which shows that

$$\lim_{x\to 0}\phi_\nu(x)=\lim_{x\to 0}\phi_\mu(x) \quad \ldots (13)$$

whenever the limit on the right exists. Hence theorem 2.

3. The following theorem asserts a necessary condition for summability (J_μ, λ):

THEOREM 3: If $\sum a_n$ is summable (J_μ, λ), then

$$a_n=o\left(\lambda_n^{\mu+\frac{1}{2}}\right) \quad \ldots (14)$$

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* We use the formula

$$J_m(x)=\frac{x^{m-n}}{2^{m-n-1}\,\Gamma(m-n)}\int_0^1 J_n(xt)\,t^{n+1}(1-t^2)^{m-n-1}\,dt \text{ for } m>n>-1.$$

Proof:—If $\sum a^n$ is summable $(J_\mu \lambda)$, then for all values of t in a certain interval excluding the origin

$$b_n J_\mu(\lambda_n t) = a_n \lambda_n^{-\mu} J_\mu(\lambda_n t) = o(1)$$

This implies that †

$$b_n = o\left(\lambda_n^{\frac{1}{2}}\right) \text{ or } a_n = o\left(\lambda_n^{\mu+\frac{1}{2}}\right)$$

4. I shall now prove a Tauberian theorem.

THEOREM 4:—If the series $\sum a_n$ is summable $\left(J_\mu^k, \lambda\right)$ for $k(\mu+\frac{1}{2}) > 1$, then it is convergent if and only if

$$\lim_{n\to\infty} \frac{\lambda_1 a_1 + \lambda_2 a_2 + \cdot \cdot + \cdot \lambda_n a_n}{\lambda_n} = 0 \quad \ldots (15)$$

Proof:—We require the following lemma.

LEMMA. (15) implies that

$$\lim_{n\to\infty} \frac{\lambda_1^2 a_1 + \lambda_2^2 a_2 + \cdot \cdot + \lambda_n^2 a_n}{\lambda_n^2} = 0 \quad \ldots (16)$$

Let the terms on the left of (15) and (16) be denoted by δ_n and η_n respectively. Now, it is easily seen that

$$\eta_n = \frac{\lambda_1^2 \delta_1 + \lambda_2(\lambda_2 \delta_2 - \lambda_1 \delta_1) + \cdot \cdot + \lambda_n(\lambda_n \delta_n - \lambda_{n-1} \delta_{n-1})}{\lambda_n^2}$$

$$= \frac{\lambda_1 \delta_1(\lambda_1 - \lambda_2) + \cdot \cdot + \lambda_{n-1} \delta_{n-1}(\lambda_{n-1} - \lambda_n) + \lambda_n^2 \delta_n}{\lambda_n^2}$$

$$= \delta_n - \frac{\sum_{1}^{n-1} \lambda_\nu \delta_\nu (\lambda_{\nu+1} - \lambda_\nu)}{\lambda_n^2} \quad \ldots (17)$$

(17) shows that (15) implies (16).

Now

$$\phi_\mu^k(t) - A_n = \sum_{1}^{n} a_\nu \{\alpha_\mu^k(\lambda_\nu t) - 1\} + \sum_{n+1}^{\infty} a_\nu \alpha_\mu^k(\lambda_\nu t)$$

$$= P_n + Q_n \text{ say} \quad \ldots (18)$$

† Sec Watson *loc. cit.* p. 650.

We shall show that if (16) is true P_n and Q_n can be made to to tend to zero.

$$Q_n = \sum_{n+1}^{\infty} a_\nu \, \alpha_\mu^k (\lambda_\nu t)$$

$$= \sum_{n+1}^{\infty} \lambda_\nu^2 a_\nu \frac{\alpha_\mu^k (\lambda_\nu t)}{\lambda_\nu^2} = t^2 \sum_{n+1}^{\infty} \lambda_r^2 a_r \, \gamma(\lambda_r t) \quad \text{say} \qquad \ldots (19)$$

Then

$$Q_n = -(\lambda_n t)^2 \eta_n \gamma(\lambda_{n+1} t) + \sum_{n+1}^{\infty} (\lambda_r t)^2 \eta_r \{\gamma(\lambda_r t) - \gamma(\lambda_{r+1} t)\}$$

$$= o(\lambda_n t)^{-k(\mu+\frac{1}{2})} + o\left\{\sum_{n+1}^{\infty} (\lambda_r t)^2 \int_{\lambda_r t}^{\lambda_{r+1} t} \left|\frac{d}{dx}\gamma(x)\right| dx\right\}$$

$$= o(\lambda_n t)^{-k(\mu+\frac{1}{2})} + o\left\{\sum_{n+1}^{\infty} (\lambda_r t)^2 \int_{\lambda_r t}^{\lambda_{r+1} t} \frac{dx}{x^{k(\mu+\frac{1}{2})+2}}\right\}$$

$$= o(\lambda_n t)^{-k(\mu+\frac{1}{2})} + o\left\{\int_{\lambda_n t}^{\infty} \frac{dx}{x^{k(\mu+\frac{1}{2})}}\right\}$$

$$= o(\lambda_n t)^{-k(\mu+\frac{1}{2})} + o(\lambda_n t)^{-k(\mu+\frac{1}{2})+1} = o(1) \qquad \ldots (20)$$

provided $k(\mu+\frac{1}{2}) > 1$.

Also

$$P_n = \sum^{n} a_r \left(\alpha_\mu^k (\lambda_r t) - 1\right)$$

$$= t^2 \sum^{n} \lambda_r^2 a_r \left(\frac{\alpha_\mu^k (\lambda_r t) - 1}{\lambda_r^2 t^2}\right)$$

$$= t^2 \sum^{n} \lambda_r^2 a_r \bar{\gamma}(\lambda_r t) \quad \text{say} \qquad \ldots (21)$$

Then

$$P_n = t^2 \sum^{n-1} \lambda_r^2 \eta_r \{\bar{\gamma}(\lambda_r t) - \bar{\gamma}(\lambda_{r+1} t)\} + t^2 \lambda_n^2 \eta_n \bar{\gamma}(\lambda_n t) \qquad \ldots (22)$$

If $\lambda_n \to \infty$ and $t \to 0$ in such a way that $\lambda_n t$ is finite and is sufficiently near the origin, we see that the second expression on the right of (22) is $o(\lambda_n t)^2$ since $\eta_n = o(1)$. We shall show that the first expression is also of the same type. In fact it is equal to

$$= -t^2 \sum \lambda_r^2 \eta_r \int_{\lambda_r}^{\lambda_{r+1}} \frac{d}{dx}\bar{\gamma}(xt)\, dx$$

$$= -\sum (\lambda_r t)^2 \eta_r \int_{\lambda_r t}^{\lambda_{r+1} t} \bar{\gamma}'(y)\, dy \qquad \ldots (23)$$

Now

$$\bar{\gamma}'(y) = -\frac{2}{y^3}\left(\alpha_\mu^k(y) - 1\right) + \frac{1}{y^2}\frac{d}{dy}\left\{\alpha_\mu^k(y)\right\}$$

$$= -\frac{2}{y^3}\left(\alpha_\mu^k(y) - 1\right)$$

$$= \frac{k\left\{2^\mu\,\Gamma(\mu+1)\right\}^k J_\mu^{k-1}(y)\,J_{\mu+1}(y)}{y^{k\mu+2}}, \text{ by } (11)$$

$$= o\left(\frac{1}{y}\right) \qquad \ldots (24)$$

in the neighbourhood of the origin. Using (24) in (23) we see that

$$P_n = o\left\{\sum (\lambda_r t)^2\,|\delta_r| \int_{\lambda_r t}^{\lambda_{r+1} t} \frac{dy}{y}\right\}$$

$$= o\left\{t^2 \sum \lambda_r\,|\delta_r|\,(\lambda_{r+1} - \lambda_r)\right\}$$

$$= o\,(\lambda_n t)^2 \text{ on account of (16).} \qquad \ldots (25)$$

Allowing λ_n and t to vary as in the remark after (22), we see from (20), (22) and (25) that

$$P_n + Q_n = o\,(1).$$

It follows therefore from (18) that

$$\lim_{n=\infty} A_n = \lim_{t\to 0} \phi_\mu^k(t)$$

whenever the limit on the right exists. This proves the sufficiency of the condition (15) for theorem 4.

The necessity of the condition (15) for convergence of $\sum a_n$ is a well known result. In fact a convergent series is summable $(R, \lambda, 1)$ and hence

$$\delta_n = \sum^n d_\nu - \sum^n a_\nu\left(1 - \frac{\lambda_\nu}{\lambda_n}\right) \to 0.$$

4·1. THEOREM 5:—If $\lambda_n a_n = O\,(\lambda_n - \lambda_{n-1})$ ... (26)
then $\sum a_n$ cannot be summable (J_μ, λ) without being convergent.

Proof:—The hypothesis implies that the series for $\phi_\mu(t)$ converges absolutely for all $t > 0$ provided $\mu > -\frac{1}{2}$. Now consider the function

$$F(t) = \frac{1}{(4t)^\mu\,\Gamma(\mu+1)} \int_0^\infty \rho^{2\mu+1} e^{-\frac{\rho^2}{4t}} \phi_\mu(\rho)\,d\rho \qquad \ldots (27)$$

The integral on the right converges for $t > 0$ and $\mu > -\frac{1}{2}$.

Also

$$F(t) = \frac{1}{(4t)^{\mu}\,\Gamma(\mu+1)} \int_0^{\infty} x^{2\mu+1} e^{-\frac{x^2}{4t}} \phi_\mu(x)\,dx$$

$$= \frac{1}{(2t)^{\mu}} \sum_1^{\infty} \frac{a_n}{\lambda_n^{\mu}} \int_0^{\infty} x^{2\mu+1} e^{-\frac{x^2}{4t}} J_\mu(\lambda_n x)\,dx$$

$$= \sum a_n e^{-\lambda_n^2 t} = \sum a_n e^{-\mu_n t} \text{ say} \quad \dots (28)$$

Term by term integration is permissible on account of absolute convergence on both sides. Now it is a simple Abelian result that

$$\lim_{t\to 0} F(t) = \lim_{x\to 0} \phi_\mu(x) \quad \dots (29)$$

whenever the limit on the right exists. Again (26) implies and is implied by

$$a_n = O\left(\frac{\mu_n - \mu_{n-1}}{\mu_n}\right) \quad \dots (30)$$

for

$$\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \leqslant \frac{(\lambda_n - \lambda_{n-1})(\lambda_n + \lambda_{n-1})}{\lambda_n^2} = \frac{\mu_n - \mu_{n-1}}{\mu_n} \leqslant 2\,\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}.$$

Now theorem 5 easily follows from (30) and (29) on account of a well known Tauberian Theorem on Dirichlet's series.

4·2. It has since been proved by K. Chandrasekaran that if a series with bounded partial sums is summable (J_μ, λ) for $\mu > \frac{1}{2}$, then it is summable $(R\ \lambda\ k)$ for every $k > 0$ to the same sum. This is proved by Wiener's method.

GLEANING

At first are developed majestically the Variations of the Velocities, then from one side enter the Equations of State, from the other the Equations of Motion in a Central Field; ever-higher sweeps the chaos of Formulae; suddenly are heard the four words: "Put $n=5$" The evil spirit V (the relative velocity of two molecules) vanishes and the dominating figure in the bass is suddenly silent; that which had seemed insuperable being overcome as if by a magic stroke. There is no time to say why this or why that substitution was made; who cannot sense this should lay the book aside, for Maxwell is no writer of programme music, who is obliged to set the explanation over the score. Result after result is given by the pliant formulae, till, as unexpected climax, comes the Heat Equilibrium of a heavy gas: the curtain then drops.

BOLTZMANN on Maxwell's gas Equations

From *The Maxwell Commemoration volume*

ORTHOCENTRE OF A CYCLIC POLYGON

BY

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1. A polygon formed by $(n-1)$ of the vertices of a n-gon (*ie* a polygon of n vertices) is for convenience, called the "face" of the n-gon opposite the omitted vertex.

We can easily prove that in a cyclic quadrangle the lines joining the vertices to the orthocentres of the opposite faces – which are triangles – meet in a point which we may define as "the orthocentre of the cyclic quadrangle." We can similarly define the orthocentre of a pentagon as the point in which the lines joining the vertices to the orthocentres of the opposite faces can be shown to concur, and so on for cyclic n-gon.

2. Let O, G, H be respectively the circumcentre, the centroid and the orthocentre of a cyclic n-gon. We shall show that they are collinear and that $HO : HG = n : 2$.

Let the result be true for a cyclic polygon of $(n-1)$ vertices. If h_r and g_r be respectively the orthocentre and the centroid of the face opposite a vertex, A_r, of the n-gon, then by our assumption, O, g_r and h_r lie on a line such that $h_r O : h_r g_r = (n-1) : 2$.

Let $A_r h_r$ meet OG in H. Then, from the triangle $O g_r G$,

$$\frac{O h_r}{h_r g_r} \cdot \frac{g_r A_r}{A_r G} \cdot \frac{GH}{HO} = -1 = \frac{n-1}{2} \cdot \frac{n}{n-1} \cdot \frac{GH}{HO}$$

$$\therefore \quad \frac{HG}{HO} = \frac{2}{n}$$

Hence H is a fixed point through which $A \ h_r$ passes, $r = 1, 2 \cdots n$, *ie* H is the orthocentre of n-gon. Thus the result holds good for the n-gon if it does so for $(n-1)$-gon. But it is true for $n = 3$, hence for $n = 4, 5, \ldots$ etc.

3. Further, with the same notations, we have from the triangle $h_r A_r g_r$

$$\frac{h_r H}{H A_r} \cdot \frac{A_r G}{G g_r} \cdot \frac{g_r o}{O h_r} = -1 = \frac{h_r H}{A_r H} \frac{n-1}{1} \left(-\frac{n-3}{n-1}\right)$$

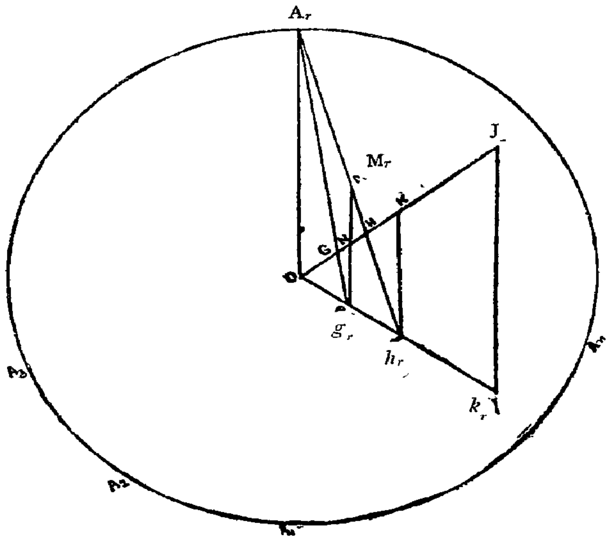
$$\therefore \quad \frac{h_r \mathrm{H}}{H A_r} = \frac{1}{n-3}$$

Therefore if K be a point on OH produced beyond H such that $OH : HK=(n-3) : 1$, then $K\,h_r=R/(n-3)$, where R is the radius of the circumcircle of the polygon, so that h_r lies on the circle with K as centre and radius $R/(n-3)$, $(r=1, 2, \ldots . n)$

We shall call this circle through the orthocentres of the faces the "orthocentric circle" of the cyclic polygon.

It is easily seen that the lines joining the vertices of the cyclic n-gon with the centres of the orthocentric circles of the opposite faces meet in the centre of the orthocentric circle of the n-gon.

4. Let k_r be the centre of the orthocentric circle of the face opposite A_r, and J be a point on OK produced such that $OK : KJ =(n-4) : 1$. Then, as that $A_r\,K : K\,k_r=(n-4) : 1$ we get $J\,k_r=\dfrac{R}{n-4}$. But the orthocentric circle of the face opposite A_r is of radius $\dfrac{R}{n-4}$; and hence it passes through J. *Thus the orthocentric circles of the faces pass through a fixed point J on OH.*



5. Returning to the orthocentre of the n-gon, let M_r be a point on the orthocentric distance of A_r such that $H\,M_r : H\,A_r=1 : (n-1)$. If $M_r\,g_r$ meets OH in N, it is easy to show that $NM_r=g_r\,N$ and that N is a fixed point being on OG so that $OG : GN :: (n-1) : 1$. Hence $N\,g_r=R/(n-1)$ and therefore M_r and g_r lie on a circle with N

as centre and radius equal to $R/(n-1)$. Calling M_r as the "point of $(n-1)$-section" of the orthocentric distance of A_r, we get that:

The points of $(n-1)$-section of the orthocentric distances of the vertices of a n-gon lie on a circle passing through the centroids of the faces.

Let us call this circle through the centroids of the faces as the centroidal circle of the n-gon.

Let n_r be the centre of the centroidal circle of the face opposite A_r, then it can be easily shown that $A_r N$ meets $O\ g_r$ in n_r and that $A_r N : N n_r = (n-2) : 1$. But $ON : NH = (n-2) : 1$. Therefore $H\ n_r = \dfrac{R}{n-2}$.

$\therefore$ As the centroidal circle of the face opposite A_r is a circle with centre n_r and radius equal to $R/(n-2)$, it passes through H, the orthocentre of the n-gon. *Thus the centroidal circles of the faces meet at the orthocentre of the n-gon.*

Before concluding, it may be interesting to consider the special case of a cyclic quadrangle.

It is evident that the "orthocentric circle" of the quadrangle has the same radius as the circumcircle. Also, as the line joining the vertices to the orthocentres of the opposite faces are bisected at the orthocentre of the quadrangle, the rectangular hyperbola passing through the vertices of the quadrangle will have its centre at the orthocentre of the quadrangle. Evidently the pedal line of any vertex of quadrangle with respect to the triangle formed by the other three passes through the orthocentre of the quadrangle.

Further if A_1, A_2, A_3, A_4 be the vertices of the quadrangle and H_1, H_2, H_3, H_4 be respectively the orthocentres of the faces opposite them, the quadrangles $A_1\ A_2\ A_3\ A_4$ and $H_1\ H_2\ H_3\ H_4$ are symmetrically related and have the same point as their orthocentre.

ON B. R. VENKATARAMAN'S CHAIN OF THEOREMS IN CIRCLE GEOMETRY.[1]

BY

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AND

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1. Introduction: Mr. B. R. Venkataraman has obtained the following results in circle Geometry[2]:—

If $2 \leqslant n \leqslant 7$, and we take n circles whose centres are concyclic, we associate with each even set a circle, and with each odd set a point in the following manner:—

With two circles, we associate their radical axis.

The radical axes of 3 circles taken in pairs are concurrent at the radical centre of the three circles. We call the radical centre the point associated with the three circles.

If we take four circles whose centres are concyclic, the four points associated with sets of three of these lie on a circle. We call the latter the circle associated with the four circles.

And so on up to $n=7$.

Mr. Venkataraman has employed the space representation of circles to obtain his results. And he does not settle whether the chain can be extended indefinitely.

The object of this paper is to show that the chain does continue indefinitely. The proof will be based on the well-known Miquel Clifford chain of theorems. Incidentally another chain of theorems is obtained—analogous to the centre-circle chain associated with the Miquel Clifford configuration.

2. Take four circles whose centres 1, 2, 3, 4 are concyclic. Let $i\,j$ be the radical axis of the circles centres i and j. And let $i\,j\,k$ be the radical centre of the circles centres i, j, k.

1. We are indebted to Prof. Narasinga Rao for drawing our attention to this problem.

2. On a new chain of theorems in circle-geometry: *Journal of the Indian Mathematical Society.* New Series Vol. 5 No. 1.

Consider the quadrangle whose vertices are 123, 124, 134, 234. The sides of this quadrangle are perpendicular to the corresponding sides of the quadrangle whose vertices are 1, 2, 3, 4, since these are respectively the radical axes and lines of centres of pairs of circles. Thus the angles of the latter quadrangle are the supplements of the angles of the former. And hence this latter quadrangle being concyclic implies that the points. 123, 124, 134, 234 are themselves concyclic. Call this circle 1234.

Now consider the Miquel Clifford configuration generated by the lines 12, 13, 14, 15. This gives that the circles 1234, 1235, 1245, 1345 all pass through a point. Starting similarly from the lines 21, 23, 24, 25 we obtain that the circles 1234, 1235, 1245, 2345 all pass through a point (*ie*) all the circles 1234, 1235, 1245, 1345, 2345, pass through a point (say) 12345.

Next consider the Miquel Clifford configuration generated by the lines 12, 13, 14, 15, 16. This gives just as in the previous case that all the points 12345, etc. lie on a circle 123456.

Continuing this way, the chain is seen to be true for all n.

3. It is well-known that given four straight lines, the circum-centres of the four triangles formed from sets of three of these lie on a circle passing through the Miquel point of the four lines.

Given 5 lines, the centres of the 5 circles associated as above with sets of four of these, lie on a circle.

And so on the chain continues.

The argument in section 2 gives an analogous chain of theorems. Thus we get the following additions to Mr. Venkataraman's chain :—

Given four circles, whose centres are concyclic, we have a circle associated with them.

Given five circles, whose centres are concyclic, the centres of the five circles assocated with sets of four of these lie on a circle.

Given six circles, whose centres are concyclic, the centres of the six circles associated with sets of five of these lie on a circle.

And so on, the chain continues indefinitely.

CONGRUENCE PROPERTIES OF $\sigma(n)$, THE SUM OF THE DIVISORS OF n.

BY

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1. Let $\sigma(n)$ denote the sum of the divisors of the positive integer n. In this note I prove the congruences given below:—

$$(1\cdot1) \qquad \sigma(3n-1) \equiv 0 \pmod 3$$

$$(1\cdot2) \qquad \sigma(4n-1) \equiv 0 \pmod 4$$

$$(1\cdot3) \qquad \sigma(6n-1) \equiv 0 \pmod 6$$

$$(1\cdot4) \qquad \sigma(8n-1) \equiv 0 \pmod 8$$

These imply in turn two other congruences viz.

$$(1\cdot5) \qquad \sigma(12n-1) \equiv 0 \pmod{12}$$

$$(1\cdot6) \qquad \sigma(24n-1) \equiv 0 \pmod{24}.$$

It appears that there are no such simple congruences to other moduli. Ramanujan* has, of course, given congruence properties of $P(n)$, the partition function, and $\tau(n)$. But there is no mention of such congruences, of $\sigma(n)$, in mathematical literature. There is only the general congruence

$$(1\cdot7) \qquad \sigma(n) \equiv 0 \pmod{k}\dagger$$

for 'almost all' n and every k.

I thank Dr. R. Vaidyanathaswami who helped me in preparing this note.

2. It is well known that if $|x|<1$ then

$$(2\cdot1) \qquad \sum_{n=1}^{\infty} \sigma(n)\, x^n = \frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \cdots\cdots$$

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* Collected papers of Srinivasa Ramanujan (Cambridge (1927) *P.* 230.

† This result, due to G. N. Watson, is given in G. H. Hardy's *Ramanujan* Cambridge 1940. *P.* 167.

'Almost all' is a phrase used by Hardy and Ramanujan in the following sense. Let P be a property of a number n expressed by a proposition $P(n)$; $N(x)$ the number of numbers up to x for which $P(n)$ is false; and that $N(x)=o(x)$. Then we say 'almost all' numbers possess the property P.

We shall denote generally by t an integer. I prove the congruences by expressing the right hand side of (2·1) as being congruent, term by term, to another power series the coefficients in which can be expressed in terms of simple divisor functions.

Modulus 3.

$$\sum_{n=1}^{\infty} \sigma(n)\, x^n \equiv \frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^4}{1-x^4} - \cdots \pmod 3$$

$$\equiv \sum_{1}^{n} T(n)\, x^n \pmod 3$$

where $T(n)$ is the excess of the number of divisors of n of the form $3t+1$ over those of the form $3t-1$

But $(3n-1) = (3t-1)(3t+1)$

$\therefore \quad T(3n-1)=0.$

i.e. $\sigma(3n-1) \equiv 0 \pmod 3.$

Modulus 4.

$$\sum_{1}^{\infty} \sigma(n)\, x^n \equiv \frac{x}{1-x} + \frac{2x^2}{1-x^2} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \cdots \pmod 4.$$

Now x^{4n-1}, since $4n-1$ is odd, can be got only from the power series for

$$\frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \cdots$$

But from Jacobi's famous theorem*. The coefficient of x^{4n-1} is zero.

$\therefore \quad \sigma(4n-1) \equiv 0 \pmod 4.$

Modulus 6.

$$\sum_{1}^{\infty} \sigma(n)\, x^n \equiv \frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} - \frac{2x^4}{1-x^4} - \frac{x^5}{1-x^5} + \cdots \pmod 6.$$

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* Jacobi's theorem is $(1+2q+2q^4+\cdots)^2 = 1+4\left(\frac{q}{1-q} - \frac{q^3}{1-q^3} + \cdots\right)$ and q^{4n-1} in this product has zero for the coefficient. See for instance; Hardy and Wright *Introduction to the theory of Numbers.* (Oxford 1938) P. 257.

As before $6n-1$ is odd and hence x^{6n-1} can be got only from the power series for

$$\frac{x}{1-x} - \frac{x^5}{1-x^5} + \frac{x^7}{1-x^7} - \ldots\ldots = \sum_{1}^{\infty} A(n)\, x^n$$

where $A(n)$ is the excess of the number of divisors of n of the form $6t+1$ over those of the form $6t-1$. But $6n-1=(6t-1)\,(6t+1)$

$$\therefore \quad A\,(6n-1)=0 \ \textit{i.e.}\ \sigma\,(6n-1)\equiv 0 \pmod 6.$$

Modulus 8.

In a similar manner we have

$$\sum_{1}^{\infty} \sigma(n)\, x^n \equiv \frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} - \frac{3x^5}{1-x^5} - \frac{2x^6}{1-x^6} - \frac{x^7}{1-x^7} + \cdots \pmod 8.$$

Since $8n-1 = (8t+1)\,(8t-1)$

$\qquad\qquad = (8t+3)\,(8t-3)$,

therefore we see that $8n-1$ has as many divisors of the forms $8t+1$, $8t+3$ as of the forms $8t-1$, $8t-3$ respectively. By arguments similar to those given above we have

$$\sigma\,(8n-1)\equiv 0 \pmod 8.$$

Now $12n-1$ is of the form $3t-1$ as well as $4t-1$ so that

$$\sigma\,(12n-1)\equiv 0 \pmod{12}$$

Similarly $\quad \sigma\,(24n-1)\equiv 0 \pmod{24}.$

3. We may use Fermats' theorem to obtain similar congruences for $\sigma_a(n)$ where $\sigma_a(n)$ denotes the sum of the ath powers of the divisors of n.

For $\sigma_3\,(3n-1) = \sum_{\delta/3n-1} \delta^3 \equiv \sum_{\delta/3n-1} \delta \equiv 0 \pmod 3$

$\therefore \quad \sigma_3\,(3n-1)\equiv 0 \pmod 3.$

Similarly $\quad \sigma_2\,(4n-1)\equiv d\,(4n-1) \pmod 4$

$$\sigma_2\,(6n-1)\equiv d\,(6n-1) \pmod 6$$

$$\sigma_4\,(8n-1)\equiv d\,(8n-1) \pmod 8$$

and so on, where $d(n)$ is the number of divisors of n.

NOTES AND DISCUSSIONS

Some Integral Inequalities

1. The object of this paper is to prove certain simple integral inequalities.

1·1. THEOREM 1. If $f'(b)$ is not equal to zero, then

$$\int_a^b f(x)\,dx \gtrless (b-a)\,f(a) + \frac{1}{2}\,\frac{\{f(b)-f(a)\}^2}{f'(b)} \qquad \dots \quad (1)$$

according as $f''(x) \gtrless 0$ in (a, b) $[a<b]$.

To prove this we require the following lemma.

Lemma

$$\int_a^x \{\varphi(x)\}^{p+1} \frac{\varphi''(x)\,dx}{\{\varphi'(x)\}^2} = (p+1)\int_a^x \{\varphi(x)\}^p\,dx - \frac{\{\varphi(x)\}^{p+1}}{\varphi'(x)} + \frac{\{\varphi(a)\}^{p+1}}{\varphi'(a)} \qquad \dots \quad (2)$$

This can easily be proved by integration by parts.

Corollary. If $p > -1$,

$$\int_a^b \{\varphi(x)\}^p\,dx \gtrless \frac{1}{p+1}\left[\frac{\{\varphi(b)\}^{p+1}}{\varphi'(b)} - \frac{\{\varphi(a)\}^{p+1}}{\varphi'(a)}\right] \qquad \dots \quad (3)$$

according as $\{\varphi(x)\}^{p+1}\,\varphi''(x) \gtrless 0$ in (a, b).

In particular, taking $p=1$, we have

$$\int_a^b \varphi(x)\,dx \gtrless \frac{1}{2}\left[\frac{\{\varphi(b)\}^2}{\varphi'(b)} - \frac{\{\varphi(a)\}^2}{\varphi'(a)}\right] \qquad \dots \quad (4)$$

according as $\varphi''(x) \gtrless 0$ in (a, b).

To prove theorem 1 we take

$\varphi(x) = f(x) - f(a)$ in the inequality (4). Then we get

$$\int_a^b f(x)\,dx - (b-a)\,f(a) \gtrless \frac{1}{2}\,\frac{\{f(b)-f(a)\}^2}{f'(b)}$$

according as $f''(x) \gtrless 0$ in (a, b), which is theorem 1.

1·2. THEOREM 2.

If $f'(b)-f'(a)-(b-a)f''(a) - \cdots - \frac{(b-a)^{n-1}}{(n-1)!}f^{(n)}(a)$ is not equal to zero, then

$$\int_a^b f(x)\,dx \gtrless (b-a)f(a)+\frac{(b-a)^2}{2!}f'(a)+\cdots+\frac{(b-a)^{n+1}}{(n+1)!}f^{(n)}(a)$$
$$+\frac{\left\{f(b)-f(a)-(b-a)f'(a)-\cdots-\frac{(b-a)^n}{n!}f^{(n)}(a)\right\}^2}{2\left\{f'(b)-f'(a)-(b-a)f''(a)-\cdots-\frac{(b-a)^{n-1}}{(n-1)!}f^{(n)}(a)\right\}} \quad \ldots \quad (5)$$

according as $f^{(n+1)}(x) \gtrless 0$ in (a, b) $(n=1, 2, \ldots)$.

To prove this take

$$\varphi(x)=f(x)-f(a)-(x-a)f'(a)-\cdots-\frac{(x-a)^n}{n!}f^{(n)}(a)$$

in (4). Then we have

$$\int_a^b f(x)\,dx-\left\{f(a)(b-a)+f'(a)\frac{(b-a)^2}{2!}+\cdots+f^{(n)}(a)\frac{(b-a)^{n+1}}{(n+1)!}\right\}$$
$$\gtrless\frac{1}{2}\,\frac{\left\{f(b)-f(a)-(b-a)f'(a)-\cdots-\frac{(b-a)^n}{n!}f^{(n)}(a)\right\}^2}{f'(b)-f'(a)-(b-a)f''(a)-\cdots-\frac{(b-a)^{n-1}}{(n-1)!}f^{(n)}(a)},$$

according as

$$\frac{d^2}{dx^2}\left\{f(x)-f(a)-(x-a)f'(a)-\cdots-\frac{(x-a)^n}{n!}f^{(n)}(a)\right\}\gtrless 0 \quad \ldots \quad (6)$$

in (a, b). We can show that (6) is satisfied when $f^{(n+1)}(x) \gtrless 0$ in (a, b). Let us suppose, for instance, that $f^{(n+1)}(x) > 0$ in (a, b). If we write

$$\varphi(x)=f(x)-f(a)-(x-a)f'(a)-\cdots-\frac{(x-a)^n}{n!}f^{(n)}(a),$$

it is clear that $\varphi(a)$, $\varphi'(a), \ldots, \varphi^{(n)}(a)$ are all zero and that $\varphi^{(n+1)}(x)=f^{(n+1)}(x)$ in (a, b). Therefore $f^{(n+1)}>0$ in (a, b) implies $\varphi^{(n)}(x)$ is monotonic increasing in (a, b), so that, since $\varphi^{(n)}(a)=0$, $\varphi^{(n)}(x)>0$ in (a, b). This together with $\varphi^{(n-1)}(a)=0$ gives $\varphi^{(n-1)}(x)>0$ in (a, b) and so on. Finally we see that $\varphi''(x)>0$ in (a, b) which proves the theorem.

2. EXAMPLES. (1) Take $f(x)=\frac{1}{1+x}$, $a=0, b=x>0$ in Theorem 1.

Then $f''(x)=\frac{2}{(1+x)^3}>0$ in $(0, x)$ so that we get

$$\log(1+x)>x-\tfrac{1}{2}x^2 \text{ if } x>0.$$

(2) Take $f(x)=\dfrac{1}{1+x}$, $a=0$, $b=x>0$ in Theorem 2.

Then we see, after a little reduction, that, if $x>0$,

$$\log(1+x) \gtrless x-\frac{x^2}{2}+\cdots+(-1)^n\frac{x^{n+1}}{n+1}+(-1)^{n-1}\frac{x^{n+2}}{2(n+1+nx)}$$

according as n is odd or even.

(3) Take $f(x)=\log(1+x)$, $a=0$, $b=x>0$, $n=1$ in Theorem 2.

Then we get, after a few reductions,

$$\log(1+x)<x/\sqrt{(1+x)} \text{ if } x>0.$$

Writing $x/(1-x)$ in the place of x we get

$$\log(1-x)>-x/\sqrt{(1-x)} \text{ if } 0<x<1.$$

(4) Take $f(x)=e^x$, $a=0$, $b=x>0$ in Theorem 2.

Then we get

$$2\,I_{n+1}\,I_{n-1}>I_n^2$$

where $I_n=e^x-\left(1+\dfrac{x}{1!}+\cdots+\dfrac{x^n}{n!}\right)$.

This may be written in the form

$$\left[I_n+\frac{x^n}{n!}-\frac{x^{n+1}}{(n+1)!}+\frac{x^n}{n!}\left\{1+\frac{x^2}{(n+1)^2}\right\}^{1/2}\right]\times$$
$$\left[I_n+\frac{x^n}{n!}-\frac{x^{n+1}}{(n+1)!}-\frac{x^n}{n!}\left\{1+\frac{x^2}{(n+1)^2}\right\}^{1/2}\right]>0.$$

The first factor is obviously positive and hence the second is also positive.

Therefore we get

$$e^x>1+\frac{x}{1!}+\cdots+\frac{x^{n-1}}{(n-1)!}+\frac{x^{n+1}}{(n+1)!}+\frac{x^n}{n!}\left\{1+\frac{x^2}{(n+1)^2}\right\}^{1/2}$$

if $x>0$.

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Some Trigonometrical Identities

1. It is wellknown that

$$4x^3-3x-1=(2x+1)^2(x-1) \quad \ldots \quad (1)$$

$$4x^3-3x+1=(2x-1)^2(x+1) \quad \ldots \quad (2)$$

In this note I use these identities to prove some trigonometrical formulae of the type

$$\left(\frac{1-2\sin\theta}{1+2\sin\theta}\right)\left(\frac{1-2\sin 3\theta}{1+2\sin 3\theta}\right)\cdots\cdots\left(\frac{1-2\sin 3^n\theta}{1+2\sin 3^n\theta}\right)=$$

$$\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\tan\left(\frac{\pi}{4}-\frac{3^{n+1}\theta}{2}\right)\times P \quad \ldots \quad (3)$$

where $P=\tan^2\left(\frac{\pi}{4}-\frac{3\theta}{2}\right)\tan^2\left(\frac{\pi}{4}-\frac{3^2\theta}{2}\right)\cdots\tan^2\left(\frac{\pi}{4}-\frac{3^n\theta}{2}\right)$ ... (4)

2. Now $\dfrac{4x^3-3x-1}{4x^3-3x+1}=\left(\dfrac{2x+1}{2x-1}\right)^2\left(\dfrac{x-1}{x+1}\right)$

If $4x_1^3-3x_1=x_2$ we then have

$$\left(\frac{x_2-1}{x_2+1}\right)\Big/\left(\frac{x_1-1}{x_1+1}\right)=\left(\frac{2x_1+1}{2x_1-1}\right)^2 \quad \ldots \quad (5)$$

Similarly defining $x_r=4x_{r-1}^3-3x_{r-1}$ and so on, we have on writing down the relations and multiplying vertically that

$$\left(\frac{x_n-1}{x_n+1}\right)\Big/\left(\frac{x_1-1}{x_1+1}\right)=\left\{\frac{(2x_1+1)}{(2x_1-1)}\cdot\left(\frac{2x_2+1}{2x_2-1}\right)\cdots\left(\frac{2x_{n-1}+1}{2x_{n-1}-1}\right)\right\}^2 \quad \ldots \quad (6)$$

Suppose we have $x_1=2$, we then see that $x_2, x_3 \cdots$ is an increasing sequence and $x_n\to\infty$ for large n so that $\frac{1}{x_n}$ may be neglected. We may thus approximate to the square roots of integers.

$$x_1=2;\ \sqrt{3}=\frac{5}{3}\cdot\frac{53}{51}\cdot\frac{140653}{140651}\ \text{(approx.)} \quad \ldots \quad (7)$$

$$x_1=4;\ \sqrt{(5/3)}=\frac{9}{7}\cdot\frac{469}{467}\cdot\frac{100925029}{100925027}\ \text{(approx.)} \quad \ldots \quad (8)$$

3. $4x^3-3x-1=(2x+1)^2(x-1)$,

$$\therefore\ \frac{1-x_2}{1-x_1}=(1+2x_1)^2.$$

As before writing n such columns and multiplying them we have

$$(1+2x_1)^2(1+2x_2)^2\cdots(1+2x_n)^2=\left(\frac{1-x_{n+1}}{1-x_1}\right) \quad \ldots \quad (9)$$

If $x_1=\cos\theta$, $x_2=\cos 3\theta$ and in general $x_r=\cos 3^{r-1}\theta$ so that we have

$$(1+2\cos\theta)(1+2\cos 3\theta)\cdots(1+2\cos 3^{n-1}\theta)=\left\{\frac{1-\cos 3^n\theta}{1-\cos\theta}\right\}^{1/2}$$

$$=\sin\frac{3^n\theta}{2}\Big/\sin\frac{\theta}{2}, \quad \ldots \quad (10)$$

the positive sign being taken, because for θ sufficiently small, the sign is positive, and both sides are polynomials in $\cos\theta$.

Similarly starting from the equation $4x^3-3x+1=(2x-1)^2(x+1)$ we get

$$(1-2\cos\theta)(1-2\cos 3\theta)\cdots(1-2\cos 3^{n-1}\theta)=\cos\frac{3^n\theta}{2}\Big/\cos\frac{\theta}{2}\ldots \quad (11)$$

Differentiating (10) and (11) logarithmically with respect to θ we have

$$\frac{\sin\theta}{1+2\cos\theta}+\frac{3\sin 3\theta}{1+2\cos 3\theta}+\cdots+\frac{3^{n-1}\sin 3^{n-1}\theta}{1+2\cos 3^{n-1}\theta}=$$

$$\frac{1}{4}\left(\cot\frac{\theta}{2}-3^n\cot\frac{3^n\theta}{2}\right) \quad \ldots \quad (12)$$

and

$$\frac{\sin\theta}{1-2\cos\theta}+\frac{3\sin 3\theta}{1-2\cos 3\theta}+\cdots+\frac{3^{n-1}\sin 3^{n-1}\theta}{1-2\cos 3^{n-1}\theta}=$$

$$\frac{1}{4}\left(3^n\tan\frac{3^n\theta}{2}-\tan\frac{\theta}{2}\right) \quad \ldots \quad (13)$$

Adding (12) and (13) we have

$$\frac{\sin\theta}{1-4\cos^2\theta}+\frac{3\sin 3\theta}{1-4\cos^2 3\theta}+\cdots+\frac{3^{n-1}\sin 3^{n-1}\theta}{1-4\cos^2 3^{n-1}\theta}=$$

$$\frac{1}{4}(\cot\theta-3^n\cot 3^n\theta) \quad \ldots \quad (14)$$

But from (12), $\frac{1}{4}(\cot\theta-3^n\cot 3^n\theta)=\sum_1^n\frac{3^{r-1}\sin 2\cdot 3^{r-1}\theta}{1+2\cos 2\cdot 3^{r-1}\theta}$ and hence

$$\frac{\sin\theta}{1-4\cos^2\theta}+\frac{3\sin 3\theta}{1-4\cos^2 3\theta}+\cdots+\frac{3^{n-1}\sin 3^{n-1}\theta}{1-4\cos^2 3^{n-1}\theta}=$$

$$\frac{\sin 2\theta}{1+2\cos 2\theta}+\frac{3\sin 2\cdot 3\theta}{1+2\cos 2\cdot 3\theta}+\cdots+\frac{3^{n-1}\sin 2\cdot 3^{n-1}\theta}{1+2\cos 2\cdot 3^{n-1}\theta} \quad \ldots \quad (15)$$

Subtracting (12) from (13) we have

$$\frac{\sin 2\theta}{1-4\cos^2\theta}+\frac{3\sin 2\cdot 3\theta}{1-4\cos^2 3\theta}+\cdots+\frac{3^{n-1}\sin 2\cdot 3^{n-1}\theta}{1-4\cos^2 3^{n-1}\theta}=$$

$$\tfrac{1}{4}(3^n\operatorname{cosec} 2\cdot 3^n\theta-\operatorname{cosec} 2\theta) \quad \ldots \quad (16)$$

4. $1+3x-4x^3=(1-x)\ (1+2x)^2$

If $x_2=3x_1-4x_1^3$, and in general $x_r=3x_{r-1}-4\ x_{r-1}^3$

then $\dfrac{1+x_2}{1-x_1}=(1+2x_1)^2.$ ... (17)

Similarly $\dfrac{1-x_2}{1+x_1}=(1-2x_1)^2.$ ... (18)

Dividing one by the other we obtain

$$\frac{(1-x_1)(1-x_2)}{(1+x_1)(1+x_2)}=\left(\frac{1-2x_1}{1+2x_1}\right)^2.$$

As before writing n such equations and multiplying we have

$$\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_n}{1+x_n}\right)\left[\left(\frac{1-x_2}{1+x_2}\right)\left(\frac{1-x_3}{1+x_3}\right)\cdots\left(\frac{1-x_{n-1}}{1+x_{n-1}}\right)\right]^2=$$

$$\left\{\left(\frac{1-2x_1}{1+2x_1}\right)\left(\frac{1-2x_2}{1+2x_2}\right)\cdots\cdots\left(\frac{1-2x_{n-1}}{1+2x_{n-1}}\right)\right\}^2 \qquad \ldots (19)$$

If $x_1=\sin\theta$, then $x_r=\sin 3^{r-1}\theta$, so that

$$\left(\frac{1-\sin\theta}{1+\sin\theta}\right)\left(\frac{1-\sin 3^{n-1}\theta}{1+\sin 3^{n-1}\theta}\right)\left[\frac{1-\sin 3\theta}{1+\sin 3\theta}\cdots\cdots\frac{1-\sin 3^{n-2}\theta}{1+\sin 3^{n-2}\theta}\right]^2=$$

$$\frac{(1-2\sin\theta)}{(1+2\sin\theta)}\,\frac{(1-2\sin 3\theta)}{(1+2\sin 3\theta)}\cdots\cdots\frac{(1-2\sin 3^{n-2}\theta)}{(1+2\sin 3^{n-2}\theta)} \qquad \ldots (20)$$

which on simplification gives

$$\left(\frac{1-2\sin\theta}{1+2\sin\theta}\right)\left(\frac{1-2\sin 3\theta}{1+2\sin 3\theta}\right)\cdots\left(\frac{1-2\sin 3^n\theta}{1+2\sin 3^n\theta}\right)=$$

$$\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\tan\left(\frac{\pi}{4}-\frac{3^{n+1}\theta}{2}\right)\times P \qquad \ldots (21)$$

where P is given by (4).

Instead of writing the results successively, we may write as follows

$$\frac{(1-x_1)(1-x_2)}{(1+x_1)(1+x_2)}=\left(\frac{1-2x_1}{1+2x_1}\right)^2$$

$$\frac{(1-x_3)(1-x_4)}{(1+x_3)(1+x_4)}=\left(\frac{1-2x_3}{1+2x_3}\right)^2$$

. . . .

$$\frac{(1-x_{2n-1})(1-x_{2n})}{(1+x_{2n-1})(1-x_{2n})}=\left(\frac{1-2\ x_{2n-1}}{1+2\ x_{2n-1}}\right)^2$$

Multiplying we have

$$\frac{(1-x_1)(1-x_2)\cdots(1-x_{2n})}{(1+x_1)(1+x_2)\cdots(1+x_{2n})}=\left[\frac{(1-2x_1)(1-2x_3)\cdots(1-2x_{2n-1})}{(1+2x_1)(1+2x_3)\cdots(1+2x_{2n-1})}\right]^2$$

... (22)

Substituting for x_1, x_2 · · · etc. we obtain

$$\left[\frac{(1-\sin\theta)(1-\sin 3\theta)\cdots(1-\sin 3^{2n-1}\theta)}{(1+\sin\theta)(1+\sin 3\theta)\cdots(1+\sin 3^{2n-1}\theta)}\right]^{1/2}$$

$$=\frac{(1-2\sin\theta)(1-2\sin 3^2\theta)\cdots(1-2\sin 3^{2n-2}\theta)}{(1+2\sin\theta)(1+2\sin 3^2\theta)\cdots(1+2\sin 3^{2n-2}\theta)}$$

$$\text{i.e. } \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\tan\left(\frac{\pi}{4}-\frac{3\theta}{2}\right)\cdots\tan\left(\frac{\pi}{4}-\frac{3^{2n-1}\theta}{2}\right)$$

$$=\frac{(1-2\sin\theta)(1-2\sin 3^2\theta)\cdots(1-2\sin 3^{2n-2}\theta)}{(1+2\sin\theta)(1+2\sin 3^2\theta)\cdots(1+2\sin 3^{2n-2}\theta)} \quad \ldots \quad (23)$$

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A Theorem on Divisibility

1. THEOREM: "*A number (represented in a scale whose radix is 'r') containing $(p-1)$ digits which are in arithmetic progression is divisible by 'p', if p be a prime and prime to r, and $r-1$, and if the common difference of the A. P. be a multiple of p.*"

Proof:—Let the number of $(p-1)$ digits be

$$(a+\overline{p-2}\,d)\,r^{p-2}+(a+\overline{p-3}\,d)\,r^{p-3}\cdots\cdots(a+d)\,r+a,$$

where $a+(p-2)d<r$, and a, d and r are positive integers. This is clearly an Arthmetico-Geometric series. Let us denote it by $f(r)$. Then

$$f(r)=\frac{a}{1-r}+\frac{dr\,(1-r^{p-2})}{(1-r)^2}-\frac{(a+\overline{p-2}\,d)r^{p-1}}{1-r}$$

$$=\frac{a(r^{p-1}-1)}{r-1}-\frac{dr(r^{p-1}-1)}{(r-1)^2}+\frac{(p-1)\,dr^{p-1}}{r-1} \quad \ldots \quad (1)$$

$$\therefore\quad (r-1)^2 f(r)=a(r-1)\,(r^{p-1}-1)-dr(r^{p-1}-1)+(p-1)\,(r-1)dr^{p-1}$$

$$=\{(a-d)\,r-a\}\,(r^{p-1}-1)+(r-1)\,(p-1)\,dr^{p-1} \quad \ldots \quad (2)$$

Now since p is prime and is prime to r, $r^{p-1}-1$ is a multiple of p (Fermat's Theorem)

Since d is a multiple of p, $(p-1)\,(r-1)\,dr^{p-1}$ is a multiple of p. Hence the whole expression on the right hand side is divisible by p.

Hence $(r-1)^2 f(r)$ is a multiple of p. But $r-1$ is prime to p. Therefore $f(r)$ is divisible by p. Hence the theorem.

1·1· *Remarks.*

(i) The number formed by repeating the $(p-1)$ digits in the same order or the reverse, any number of times is also divisible by p.

Thus the number $16es$ (e and s denote 11 and 16.) in the scale of 17 is divisible by 5 by the above theorem. Hence the number $16es16esse6116es \cdots\cdots$ ($16es$ being repeated any number of times in the same order or the reverse, is also divisible by 5).

(ii) If a and d are both multiples of p, then $f(r)$ is divisible by p for all values of p and r.

1·2 *Corollary*: A number (in scale r) containing $k(p-1)$ equal digits is divisible by p if p is prime and prime to r and to $r-1$.

Proof: If in the expression $a+(a+d)r \cdots\cdots (a+p-2d)r^{p-2}$, we put $d=0$, it becomes $a+ar+ar^2 \cdots\cdots ar^{p-2}$ whose sum is $a(r^{p-1}-1)/(r-1)$. Hence $(r-1)\phi(r)=a(r^{p-1}-1)$. As before $r^{p-1}-1$ is divisible by p since p is prime and prime to r. Hence $(r-1)\phi(r)$ is a multiple of p. But $(r-1)$ is prime to p. Therefore $\phi(r)$ is divisible by p. Also the $(p-1)$ digits may be repeated any number of times. Hence the number may contain $k(p-1)$ digits, k being a positive integer.

Again we know that any polynomial $\phi(r)$ (in radix r) is divisible by $r+1$ if $\phi(-1)=0$. Since in the above number the digits are equal, $\phi(-1)=a-a+a \cdots\cdots$ to $p-1$ terms. This is equal to zero because $p-1$ is even (p is odd except when $p=2$). Hence it is divisible by $p(r+1)$ also. Hence the number is divisible by $p(r+1)$ if p is prime to $(r+1)$. In the common scale since 11 is a prime, it is prime to any other prime number. Hence if a number contains $k(p-1)$ equal digits, it is divisible by 11 p where p is prime and prime to 10 and 9 in the ordinary scale.

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A Note on Four-point Systems of Conics

The object of this note is to show that certain properties of conics may be proved by a simple device explained below:

1. THEOREM: *If S_1, S_2, S_3 be three conics and P, Q be two conics through the intersections of S_1, S_2 and S_1, S_3 respectively, then a conic R passes through the intersections of S_2, S_3 and P, Q.*

Let the cartesian equations to the conics S_1, S_2, S_3, P, Q be respectively

$S_1=0$, $S_2=0$, $S_3=0$, $S_1+\lambda S_2=0$, $S_1+\mu S_3=0$. The coordinates of the points which satisfy the last two equations will also satisfy $\lambda S_2-\mu S_3=0$.

But the latter is the equation to a conic through the intersections of S_2 and S_3, and thus the theorem is proved.

Further, if $S_1+\lambda S_2=0$ and $S_1+\mu S_3=0$ be circles, then $\lambda S_2-\mu S_3=0$ is also a circle. Therefore we have

2. THEOREM: *If circles can be drawn through the intersections of conics S_1, S_2 and S_1, S_3 then a circle can be drawn through the intersections of S_2, S_3 and the three circles will be coaxal.*

Again, let $S_2\equiv T^2$, $S_3\equiv T'^2$, where T and T' are linear expressions. Then the equation to the conic R will be $\lambda T^2-\mu T'^2=0$.

But this equation represents the pair of lines $\sqrt{\lambda}T+\sqrt{\mu}T'=0$; $\sqrt{\lambda}\,T-\sqrt{\mu}\,T'=0$
which form a harmonic pencil with the lines $T=0$, $T'=0$.

Thus we get the

3. THEOREM: *If P, Q be two conics having double contact with the conic $S=0$ at the points where the latter is intersected by the lines T and T' respectively, then the intersection of P and Q lie on two lines which divide T and T' harmonically.*

It is clear that similar results can be established for conicoids.

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Omar Khayyam's Calendar

Omar Khayyam, whose Rubiyat has given him in the west the greatest following of admirers ever enjoyed by any orientalist, was also the greatest mathematician and astronomer of his age. Born at Nishapur in Persia, he flourished during the latter half of the eleventh century. His mathematical achievements consist of the general geometrical solution of the cubic equation and the "Calendar" known as Tarikh-i-Jalal. Of his published works the following two books are of special significance :—

(1) Jabr-o-Moqaballa—which contains his contributions to Algebra.

(2) Zeecha-i-Malik Shahi—which contains his work on the Calendar.

The second book is not available now, and in the present note I shall present what information I could glean about his Calendar from different sources.

Among the Muslims the purely lunar Hejira Calendar is esteemed due to the following saying of the Prophet :—

"We are unlettered people and mathematical complications are unknown to us. So we adopt the simple lunar year." When the Muslims acquired a great empire and agriculture became the most important source of state revenue, the adoption of a solar Calendar became inevitable since land taxes could be collected only on a seasonal basis. The rulers were, however, reluctant to abandon the Lunar Calendar blessed by Mohammed. A compromise was reached, land revenue being collected according to the ancient Persian Solar Calendar and state expenditure being incurred according to the Lunar year. In this way the state treasury was suffering an appreciable loss, because the Lunar year is eleven days shorter than the Solar year. Even the Persian Solar year was fast deviating from the seasons since the extra day of the leap year which was given a great religious significance by the ancient Persians was suppressed to make the Persian converts forget their past religion.

This was the chaotic state of the Calendar when Malik Shah the ruler of Persia ordered Omar Khayyam to evolve a Calendar which could be used with advantage in all the financial transactions of the state. Omar's Calendar is solar. The first essential of the accuracy of any solar calendar is to know correctly the time taken by the sun to complete one round of the Ecliptic. With the most

up-to-date instruments—insignificant although they were compared to the present ones—he calculated at the Isphahan observatory the length of the solar year to be 365 days, 5 hours and 49 minutes—only 11 seconds greater than the length of the tropical year accepted at present. The new year of Khayyam's Calendar begins from the noon of the day when the sun enters the burj-i-hamal—one of the twelve parts into which he divided the celestial sphere. This is the day when day and night are equal and spring commences. It happens when the sun is at the vernal equinox. Friday the 10th of Ramadan 471 A. H. was the first day of Omar's Calendar.

He divided the year into twelve months. The first eleven months of every ordinary year consisted of 30 days each and the last consisted of 35 days. In this way an ordinary year came to 365 days. Every fourth year, the last month was given 36 days instead of 35, thus making the year of 366 days. Every thirty-second year which is a leap year if a cycle of four years is maintained, was kept an ordinary year consisting of 365 days—the last month being only of 35 days. Every thirty-third year consisted of 366 days—the last month containing 36 days only.

Thus Omar Khayyam formed a cycle of 33 years. The first eleven months of every year consisted of 30 days each. There are twenty-five ordinary years and eight leap years in one cycle.

Therefore 10,000 years of Khayyam's Calendar are equal to 3652424 mean solar days.

One tropical year = 365·2422 mean solar days.

Hence 10,000 tropical years = 3652422 mean solar days.

According to Gregorian Calendar used at present in all the advanced countries of the world 10,000 years are equal to 3652425 mean solar days.

Omar Khayyam's Calendar, therefore, gives an error of only 2 days in 10,000 years while the present Gregorian Calendar gives an error of 3 days in 10,000 years.

The names of the twelve months of Khayyam's Calendar were as follows:—

(1)	Farwar-i-Deen	(7)	Meher
(2)	Urdi Bahisht	(8)	Aban
(3)	Khoor Dad	(9)	Azar
(4)	Teer	(10)	Dee
(5)	Amar Dad	(11)	Behman
(6)	Shahr Yoor	(12)	Isfandar.

Omar Khayyam's Calendar remained in vogue during the reign of the Seljuks and the Khowarizmis. Due to their religious fanaticism the early Tartars abolished it when they came to power and reintroduced the Hejira Calendar as the only authorised Calendar of the state.

A modified form of Omar Khayyam's Calendar is still in use among the Parsi community of India.

References

(1) "Khayyam" by Sulaiman Nadvi.

(2) "Sher-ul-Ajam" by Shibli Nomani.

(3) "Chahar Maqalla" by Nizami-i-Aruzi-i-Samarqandi.

St. John's College Delhi. UMAR SULAIMAN.

A well known geometrical fallacy

The following Geometrical Fallacy which seeks to prove that a right angle is equal to an obtuse angle is given in Ball's *Mathematical Recreations and Essays* and should be of interest to students. The argument is perfectly logical and the source of error is only the wrong figure.

1. Let ABCD be a rectangle. Take any point E so that BE = BC. Join DE. Draw the perpendicular bisectors of AB and DE and let them meet in O. Join OE, OB, OD and OA.

In the triangles OBE and OAD

OE = OD and OB = OA $\because$ O lies on perp. bisectors of AB and DE.

Also BE = AD (construction).

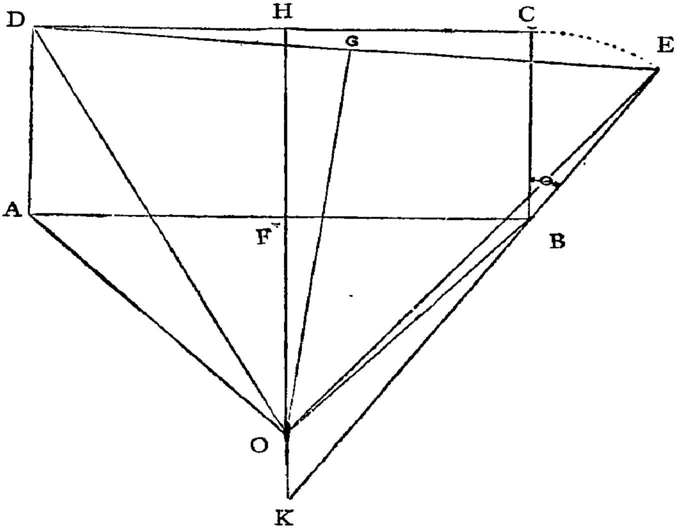
$\therefore$ Δ^s are congruent and $\angle OBE = \angle OAD$ ... (1)

but $\angle OBA = \angle OAB$ ($\because$ OA = OB); hence $\angle ABE = \angle BAD$ ... (2) the latter being a right angle and the former an obtuse angle.

Wherein lies the fallacy? We shall prove that the point O lies below the point K, which is the intersection of BE and right-bisector of AB; so that while (1) is true, angle OBA is not part of OBE and cannot be subtracted from it to get (2).

2. Let F, G and H be the midpoints of AB, DE and CD. Let EB meet FO produced in K. and let $AB=2a$, $BC=b$ and $\angle CBE=\theta$. Take FB and FH as the axes of x and y.

The coordinates of D and E are $(-a, b)$ and $(a+b \sin \theta, b \cos \theta)$ respectively.



$\therefore$ The slope of the line DE is

$$\frac{b-b \cos \theta}{-a-(a+b \sin \theta)}=\frac{-b\,(1-\cos \theta)}{2a+b \sin \theta}$$

$\therefore$ slope of the line GO, a line $\perp$ to DE, is $\dfrac{2a+b \sin \theta}{b\,(1-\cos \theta)}$.

The coordinates of G are

$$\left(\frac{-a+a+b \sin \theta}{2}, \frac{b+b \cos \theta}{2}\right) \text{ i.e. } \left(\frac{b \sin \theta}{2}, \frac{b\,(1+\cos \theta)}{2}\right).$$

$\therefore$ The line GO is

$$y-\frac{b\,(1+\cos \theta)}{2}=\frac{2a+b \sin \theta}{b\,(1-\cos \theta)}\left(x-\frac{b \sin \theta}{2}\right)$$

Put $x=0$ in the above equation, and we get the y coordinate of O, after simplification to be $-a \cot \frac{\theta}{2}$ which is negative for $0<\theta<\frac{\pi}{2}$.

Again $\frac{FK}{BF}$ = cot ∠BKF = cot θ (∵ BC ∥ HK and BE is a transversal.) ∴ FK = a cot θ.

Therefore FK is less than FO since $a \cot \theta < a \cot \theta/2$; in other words O lies on FK produced in the downward direction.

G. L. DHAMIJA.

Note on Q. 1780 *

In this short note Q. 1780, is generalised thus:—

If m is any positive integer and $M = \prod_{r=1}^{k} m_r$ where $m_r = r^{th}$ odd number which is less than m and prime to it and k = the number of such m_r's, then

$$M^4 \equiv 1 \pmod{m}$$

Proof:—(i) Let m be odd. Then if $m - m_r = n_r$, n_r is even, less than m and prime to m. Also the number of such n_r's is precisely k.

Hence $M = m_1 \cdot m_2 \cdots m_r \cdots m_k$

$$\equiv (-1)^k n_1 \cdot n_2 \cdots n_r \cdots n_k \pmod{m}$$

∴ $M^2 \equiv (-1)^k P \pmod{m}$

where P is the product of all numbers less than and prime to m.

∴ $M^4 \equiv 1 \pmod{m}$ using Gauss' extension of Wilson's Theorem.

(ii) Let m be even; then all numbers which are less than and prime to m are odd.

∴ M = P. Hence $M \equiv \pm 1 \pmod{m}$ by Gauss' Theorem.

∴ $M^4 \equiv 1 \pmod{m}$.

It is also interesting to note that in (i) (i.e.) when m is odd, $M^4 \equiv 1 \pmod{16m}$ since $M^2 + 1 \equiv 0$ (mod. 2) and $M^2 - 1 \equiv 0$ (mod. 8) and 16 is prime to m whenever it is odd.

Madras. C. S. VENKATARAMAN.

* Question 1780 (M. S. SRINIVASAN): If p be prime and $M = 1.3.5.7....(p-2)$, prove that $M^4 \equiv 1$ (mod. p). (*The Mathematics Student*, June 1941 p. 93).

A Problem of Ramanujam †

Solve

$$x^2=a+y \quad \ldots \quad \text{(i)}$$
$$y^2=a+z \quad \ldots \quad \text{(ii)}$$
$$z^2=a+u \quad \ldots \quad \text{(iii)}$$
$$u^2=a+x \quad \ldots \quad \text{(iv)}$$

1. Suppose x, y, z, u are the roots of a biquadratic

$$t^4+p_1 t^3+p_2 t^2+p_3 t+p_4=0$$

We denote $\sum x^n$ by S_n.

Now $S_1=-p_1$

$S_2=4a+S_1=4a-p_1$ from the given equations ... (a)

Also $S_2+p_1 S_1+2p_2=0$

$\therefore$ Substituting for S_1, S_2 we have $p_2=\dfrac{p_1^2+p_1-4a}{2}$ ... (b)

2. Subtract (iii) from (i) and (iv) from (ii)

$x^2-z^2=y-u$; $y^2-u^2=z-x$

$\therefore$ $(x^2-z^2)(y^2-u^2)=(y-u)(z-x)$

or $xy+zy+ux+uz=-1$

But $\sum xy=p_2$

So $xz+uy=p_2+1$ ... (c)

3. $x^3=ax+xy$

$y^3=ay+yz$

Adding $S_3=aS_1+xy+zy+ux+uz$

$=-ap_1-1$ ... (d)

Also $x^2z^2=a^2+au+ay+uy$

$y^2u^2=a^2+az+ax+zx$

Adding, $x^2z^2+y^2u^2=2a^2+aS_1+uy+zx$

$(xz+uy)^2=2a^2+aS_1+p_2+1+2p_4$ $\quad [xyzu=p_4]$

$(p_2+1)^2=2a^2-ap_1+p_2+1+2p_4$

$\therefore$ $2p_4=p_2^2+p_2-2a^2+ap_1$ ... (e)

4. Evidently

$(x^2-y^2)(y^2-z^2)(z^2-u^2)(u^2-x^2)=(x-y)(y-z)(z-u)(u-x)$

or $(x+y)(y+z)(z+u)(u+x)=1$

or $(x^2z^2+u^2y^2+2xyzu)+\sum x^2yz=1$

$(p_2+1)^2+p_1 p_3-4p_4=1$ $\quad \{\sum x^2yz=p_1 p_3-4p_4\}$

Substituting for p_4 from (e)

$p_1 p_3-p_2^2+4a^2-2ap_1=0$ ... (f)

† Vide *Collected Papers of Srinivasa Ramanujan*, 1927, p. 332 Q. 722 Also J. I. M. S. Series I Vol. VII p. 240.

5. We know

$$S_3 + p_1 S_2 + p_2 S_1 + 3p_3 = 0$$

Substituting for S_3, S_2, S_1, p_2

we get

$$6p_3 = 3p_1^2 + p_1^3 - 10\, ap_1 + 2 \qquad \ldots \quad (g)$$

Substitute for p_3 from (g) in (f)

we get

$$p_1^4 - (4a-3)\, p_1^2 - 4p_1 = 0 \qquad \ldots \quad (h)$$

$$p_1 \{p_1^3 - (4a-3)\, p_1 - 4\} = 0$$

The cubic in p_1 can be solved by the usual methods. p_1 known, p_2, p_3, p_4 can easily be discovered and the corresponding biquadratic in t can be framed. The biquadratic may be solved by usual methods for x, y, zu; For the particular value $p_1 = 0$ the biquadratic is

$$t^4 - 2a\, t^2 + \tfrac{1}{3}\, t + a^2 - a = 0$$

6. By employing the same methods, we can solve the system of equations

$$x^2 = a + y$$

$$y^2 = a + z$$

$$z^2 = a + x$$

much more rapidly than Ramanujan did. His is a very laborious method.

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ABDUS SALAM,
Fourth Year Student

Reform of the Hindu Calendar

In Prof. Hansraj Gupta's article on "Calendar" in *the Mathematics Student* Vol. X pp. 105-109, it is stated that "The Hindu Calendar needs reform in two directions. First, it should begin with the Vernal Equinox as was actually intended to. Secondly, the length of the Year should be corrected. The length of the Year as stated in the *Surya Siddhanta* is in excess of the *correct* length by nearly 24 minutes".

I fear, the reforms suggested will if carried out efface the eternal astronomical character of the Hindu Calendar, the Hindu names of months, weekdays etc. Whenever the Solar (The Indian Solar), month *Chaitra* starts (not the lunar one which starts on the New-Moon day), the *Chitra* Constellation will be on the eastern horizon at the beginning of the night, on that day. On the first of Indian Solar *Visakha* months, the star *Visakha* will rise in the East at Sun-set. Thus the names of the Hindu months have got astronomical significance and they retain this for all time. This is due to the fact that *Aswini* is a fixed star in the celestial sphere having no motion. The genius of the Hindus is revealed in this arrangement. Vernal Equinox being a moving point is quite unsuited to be the origin. We know how the Right Ascension and the declination of any star go on changing from moment to moment. If the Vernal Equinox be made the origin, the astronomical significance of the name *Chaitra* of the *Chitra* month will be lost. Thus we will be purchasing the reform with a great loss.

As regards the second reform, it is stated that the length of the Hindu Year is in excess of the correct length by nearly 24 minutes. It is not meant by this that the length of the Sidereal Year given by the Hindus is in excess of the true Sidereal Year (modern value by 24 minutes). From the value of 24 minutes, it looks that by correct year was meant the Tropical Solar Year. The Indian Year being a Sidereal one cannot but differ by that value from the Tropical Solar Year. There are three kinds of years, viz. the Tropical Solar Year, the Sidereal Year and the Anamolistic Year. The periods of time occupied by the Sun in making complete revolutions with respect to the moving point Vernal Equinox, or the fixed point Aswini star, or the moving point Perigee denote the above three years. The Hindus have deliberately chosen the sidereal year for use, as it maintains the astronomical significance of the names of the months etc., and also maintains the seasons at their proper places in the course of the Year for a great length of time.

What is a *correct year*? If by a correct year is meant that period of time which brings back the same season at the beginning of the year, we have yet to know by actual experience whether the seasons return regularly back once in 365·2422 mean solar days or once in 365·2587 mean solar days (Indian). Seasons which are changes in the meteorological conditions of places on Earth, are not purely a function of the position of the Sun with respect to the Equator. Seasons form a function of a number of complex causes

of which the position of Apogee may be also one. Further two different places having the same latitude do not have the same season at the same time. If a place in India in the western coast receives rain in the month of June, another place in India situated on the eastern coast and having the same latitude, has no rain at all at the same time. Thus the seasons depend on several causes of which the position of the Sun is one. Western astronomy has not, as yet, established the fact that the seasons correctly follow the Tropical Solar Years and fall back with respect to the Sidereal Years. The time from Pope Gregory the Great (1582), to the present time (1943), which is barely 360 years, is too short a period to come to any conclusion. But there is positive evidence as regards the Sidereal Year being a correct one, in the sense that the seasons repeat themselves in that period. We know that from the Vedic times, India has been observing the Sidereal Year, the lunar system being corrected now and then to agree with the solar calculation. In this sidereal system, when Lord Shri Krishna was born, about 5,000 years ago in the month of Sravana, Tithi Ashtami and Nakshatra Rohini, there were clouds, thunder and lightning, and rain and floods in the Jumna. Even now when we celebrate the Krishnashtami day after five thousand years, we have similar meteorological phenomena.

If the proposed reform be carried out, the distant future alone can say whether the reform is well advised; but a few minutes thought will tell us that in the course of years the Chitra month will lose its astronomical significance, as the First Point of Aries has nothing to do with Aries now. In my opinion it would be better if the existing system be allowed to remain as it is.

In the Indian system the ordinary year always began with the Sun in Aswini, but the sacrificial year began with different points on the Ecliptic according to the nature of the sacrifices i. e. Vernal Equinox, Autumnal Equinox, Chaitra New Moon, Summer solstice etc. After a very long study the Hindu Rishis have arranged the present Indian Year.

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R. KRISHNAMURTI.

Feuerbach's Theorem

The proof[1] given here depends upon the following facts :—

(i) The rectangular hyperbola inpolar[2] to a circle has its centre on the circle;

(ii) The centre locus of the rectangular hyperbolas circumscribing a given triangle is its nine point circle;

(iii) There are only two rectangular hyperbolas circumscribed to a given triangle and inpolar to a Bipunctual[3] circle, namely those passing respectively through the points for which the circle is Bipunctual[4]. Hence their centres are the common points of this circle and the ninepoint circle.

Now if the points for which a circle is Bipunctual coincide, it becomes an incircle and the two rectangular hyperbolas circumscribing the triangle inpolar to this circle then coincide also, i. e. *the nine point circle touches the incircle at the centre of the unique rectangular hyperbola, circumscribed to the triangle and inpolar to the incircle.* This hyperbola passes through the Gergonne point of the incircle, *i. e.* where the lines joining the vertices of the triangle to the points of contact of the incircle with the opposite sides meet; it also passes through the incentre *

Punjab University. SAHIB RAM MANDAN.

On Residue Chains

1. The object of this note is to prove the

THEOREM. "If m has a primitive root, then there is a chain

$$r_1, r_2, r_3, \ldots\ldots\ldots, r_{h-1}, r_1; \text{ where } h=\phi\ (m);$$

of the numbers other than $(m-1)$ which are less than and prime to m, such that the $(h-1)$ numbers

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1. The lines of this proof I owe to Late Pandit Hemraj.

2. H. F. Baker, Principles of Geometry, Vol. 2.

3. A conic is said to be Bipunctual, when it circumscribes two triangles in perspective with a given triangle and inscribed to it.

4. It is then said to be Bipunctual *w. r. t.*, the 2 centres of perspectivity. The term Bipunctual I learn from Prof. C. V. H. Rao.

* Taking the incircle as the limit of pedal circles of the triangle we get the same result.

$r_1r_2,\ r_2r_3,\ r_3r_4,\ \ldots\ldots,\ r_{h-2}r_{h-1},\ r_{h-1}r_1;$

are congruent modulo m to the numbers

$r_1,\ r_2,\ r_3,\ \ldots\ldots\ldots\ldots,\ r_{h-1};$

each to each in some order."

The proof is so simple that my first feeling was not to publish it, but then since it had taken me some time to get it, I thought it might be worth recording.

Proof. Let g be a primitive root of m.

Then the numbers: $1,\ g,\ g^2,\ g^3,\ \ldots\ldots,\ g^{h-1}$, form a complete residue set modulo m, Evidently $g^t \equiv m-1$ (mod m), where $t = h/2$.

Excluding this, the chain

$$1,\ g^{t+1},\ g,\ g^{t+2},\ g^2,\ g^{t+3},\ \ldots\ldots,\ g^{h-1},\ g^{t-1},\ 1;$$

satisfies the requirements of the theorem.

Since the number of primitive roots of m is $\phi(h)$, there are at least $\frac{1}{2}\phi(h)$ such chains.

For $m=11$, the primitive roots 2 and 7 give the chains:

1, 9, 2, 7, 4, 3, 8, 6, 5, 1; and 1, 4, 7, 6, 5, 9, 2, 8, 3, 1;

the other two primitive roots viz. 6 and 8 giving the same chains.

When m has no primitive root, Prof. Vijayaraghavan and Chowla have recently proved a similar theorem. The above theorem is intended to supplement theirs. I am not aware if their proof has been published.

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HANSRAJ GUPTA.

GLEANING

Diodorous the rhetorician, who denied the existence of motion, arguing that the motion was not in the place from where the body moved, nor in that to which it moved, and that accordingly, it did not exist at all. Diodorous once met with an accident and had a severe fall which put his shoulder out of joint and he applied to Herophilus the surgeon to set it. Herophilus asked where the fall took place, whether on the spot where the shoulder was or in that to which it fell; but the suffering philosopher entreated him to begin by setting his limb and that they would talk about the existence of motion afterwards.

E. M. SEWELL

Ancient History of Assyria, Egypt and Babylonia.

(PER. S. K. VISVESWARA SASTRI).

REVIEWS

Algebra

By F. W. Levi, Dr. Phil. Nat. Hardinge Professor of Higher Mathematics, Calcutta University.

The great advances which have been made at the present day in the study of basic mathematical ideas (like group, ring, field, partially ordered set, topology etc.) raise the problem of to what extent and in what manner they should be introduced in graduate and post-graduate college teaching. The problem is a particularly vexatious one in this country which, situated outside the highways of mathematical progress, has only recently been feeling its way into modern mathematics. The courses of study for the Honours and M. A. degrees, and the teaching, continue to be out of date in placing the main stress on the manipulative side. Any introduction of modern ideas into the courses is resisted by the teachers in the belief, widely held, that they are too difficult for regular teaching, and suitable only for the research stage. This plea is however only a half-truth. It is true that advanced ideas like 'topological structure' or 'partially ordered set' are unsuitable for inclusion in the Honours or M. A. courses of study, but this is not so much because of their intrinsic difficulty, as because of the fact that they do not stand in intimate relation with the actual topics which have to be covered in the course. But the case is quite otherwise in regard to a different class of mathematical ideas—like group, ring, field, and invariants of linear transformations—which are continually involved in the traditional topics of the course. The advantage which the student derives from the course, whether in specific mathematical ability, or in general mathematical culture lies precisely in the extent to which he has succeeded in glimpsing these ideas behind the stereotyped manipulations. Ideas of this kind cannot be difficult to communicate to the learner, if they are taught in close relationship with their applications; they would of course prove difficult if they are taught in an abstract and formal fashion from a treatise. Their successful introduction in teaching is thus primarily a matter for the teacher's technique.

Professor Levi's introduction of Modern Algebra into the regular M. Sc. course in the Calcutta University has been watched with very great interest by teachers in India, and it is a matter of general satisfaction that it has become established there and assimilated into the curriculum. The volume under review is the instructive record of the lines adopted for the teaching of the subject. As Professor Levi explains in the introduction, he has been very cautious in the matter of innovations, and has preferred to choose a middle way both in regard to content, and in regard to method. As the title 'Algebra' of the book indicates, Prof. Levi does not confine himself to the

topics of Modern Algebra, but includes also 'approximate solution of numerical equations', and 'continued fractions'. We would entirely agree with the author, that he has been judicious in 'respecting the autonomy of numerical calculation', and has avoided bringing in the notion of the 'formally real' field. The author's 'half-classical' treatment of continued fractions is interesting and apparently original. As regards method, the author explains that he has not followed the formal exposition of treatises on Modern Algebra like Van der Waerden's but has begun with the classical theory of linear equations and vector-spaces, and introduced the basic notions of modern algebra with reference to them. As I have tried to indicate, this is the method which must be adopted for the successful teaching of modern algebra, and it is the reflection of this method in the book that makes it a valuable manual for teaching. A suggestion that might be made for increasing the usefulness of the book in this regard, is the insertion of suitable exercises for disciplining the student in the wielding of the new ideas.

Apart from Chapter I which deals with the theory of linear equations, and Chapters IV and V which deal with continued fractions and approximation of roots, the remaining Chapters II, III, and VI deal with Modern Algebra proper. Chapter II deals with Rings and Fields in general, Polynomial rings and unique factorisation domains, and with Algebra and Normal extensions of fields. Chapter III deals with roots of unity, cyclotomic polynomials and their irreducibility, with the Galois fields and their applications, with the complex number field, and with symmetric polynomials and Resultants. Without going into the Galois theory of solution of an equation (which the author reserves for a second volume), the algebraic solutions of the cubic and biquadratic are given in this chapter in the spirit of Modern Algebra as illustrations of equations soluble by radicals. Chapter VI is devoted to the matrix algebra over a field, to the theory of elementary divisors, and to special types of real and complex matrices.

That the volume under review is an excellent text-book for the teaching of Modern Algebra in colleges is due primarily to the great care exercised in the choice of topics, in the relative distribution of emphasis between them, and the general orientation which the author has given them. There have appeared some recent publications attempting to inject Modern Algebra into the traditional topics of Algebra; but none of them can be said to be suitable for regular use in colleges. We warmly congratulate Professor Levi on this pioneer-work, which is apparently the only one extant, in English or any other language, suitable for use in a first course in Modern Algebra in the colleges. We hope that in the wake of this book, Modern Algebra will become popular in Indian Universities, and be prescribed as an optional subject for the Honours and M. A. examinations.

R. V.

The Lagna System of the Vedanga-Jyautisha or An Obsolete Indo-Iranian Time Measure.

By B. R. Kulkarni, Pp. 48. (1943) (Rajwade Samshodan Mandal, Dhulia.)

This brochure contains a new exposition of the idea of Lagna in Vedanga Jyautisha, which is avowedly the oldest Indian work extant in Astronomy. The Hindu Calendar, ever since its inception, has been a complicated one involving the positions, relative and absolute, of the Sun, the Moon and the stars. We should welcome all attempts at a correct understanding of this Calendar.

There are four different conventional ways in which the day may be considered to begin, viz., sun-rise, sun-set, mid-day and mid-night, though the first is, perhaps, the most natural. In the Nautical Almanac, where the different systems of calendar are explained, we find that the day was reckoned by the Babylonians from sunset for civil and religious purposes and for astronomical observations and predictions, with the exception that the famous Babylonian astronomer Kidinnu reckoned it from mid-night as a more convenient starting point for astronomical tables. In modern astronomy, the change from midday to midnight as the starting point of the astronomical day was effected on January 1st, 1925.

Mr. Kulkarni puts up evidence from ancient Jain astronomical texts that the equinoctial time was reckoned acronychally, *i.e.*, from sunset and not cosmically, *i.e.*, from sunrise. But the author himself admits that the references contain a good deal of confusion and attempts to straighten them out in his own way. The bone of contention is the semi-verse:

श्रविष्ठाभ्यां गुणाभ्यस्तान् प्राग्विलग्नान्विनिर्दिशेत् ।

which is interpreted as:

> "The 15th (or the 12th) Nakshatra from the Suryanakshatra of a day is the Lagna for the same day",

and takes us on to the Nakshatra rising at sunset.

The style is expansive and diffusive. A more concise presentation concentrated on relevant material would have carried greater conviction. But there is no doubt that the author has taken great pains and left possibly no stone unturned in establishing his theory to the best of his abilities. He deserves every encouragement.

Mysore. A. A. K.

An introduction to Pure Solid Geometry by Dr. G. S. Mahajani M.A., Ph.D. (Cantab). Fergusson College, Poona. Aryabhushan Press, Poona, 1943 Price Rs. 3/-

This booklet of just over a hundred pages is the second edition of another with the same title published in 1940 and is intended as a text book in the elements of solid geometry for the Intermediate in Arts and Science courses in Indian Universities. The topics covered include the metrical relations between points, lines and planes in space, the properties of the tetrahedron, sphere and additional mensurational formulae relating to pyramids, cones and cylinders. The last chapter is devoted to polyhedral angles, Eulers theorem and regular polyhedra. An appendix on the prismatoidal formula and a collection of miscellaneous examples bring up the rear.

Prof. Mahajani is an educationist of experience and the author of several well-written books on Analysis and Differential Geometry, and writes with a deep and intimate knowledge of where the shoe—particularly a new one like solid geometry—pinches the tender feet of the young mathematical pilgrim. Wherever possible, 3-dimensional results are correlated with the corresponding results in plane geometry, and the interest of the learner stimulated by topics like semi-regular solids and space filling patterns. The properties of the tetrahedron include the Monge point—where the planes through the mid-point of each edge perpendicular to the opposite edge meet—and are neatly derived from the circumscribing parallelopiped. The book contains just the essential propositions and illustrative examples for an introduction to solid geometry, and should prove very useful to a much wider circle than that for which it is intended.

A. N. R.

GLEANING

Any one with mathematical friends will have noticed that some of them like to work early in the morning, while others do nothing till after dark. The replies on this point indicate a curious trend. The mathematicians of the northern races prefer to work at night, while the Latins favour the morning. Among night workers prolonged concentration often brings on insomnia as they grow older and they change reluctantly to the morning. Felix Klein, who worked day and night as a young man, once indicated a possible way out of this difficulty. One of his American students complained that he could not sleep for thinking of Mathematics. "Cant sleep, eh?" Klein snorted. "What's chloral for?" However this remedy is not to be recommended indescriminately; it probably had something to do with Klein's own tragic break-down.

From E. T. BELL: *Men of Mathematics*.

ANNOUNCEMENTS AND NEWS

The following gentlemen have been admitted as members of the Indian Mathematical Society:

V. S. Krishnan Esq., M.A. Research Student, Madras University, 35 Edward Elliot Road, Mylapore, Madras,

S. Raghavachari Esq., M.A., Principal, National Tutorial College, Kumbakonam,

K. Aggarwal Esq., c/o. H. R. K. Aggarwal Esq., Majithmundi Amritsar,

K. Chandrasekharan Esq., M.A., M.Sc., Lecturer, Andhra University, Guntur,

S. S. Sankaranarayanan Esq., Student, B.A. (Hons) Class, 119 Big Street, Triplicane, Madras,

S. Minakshisundaram Esq., M.A., D.Sc., Lecturer, Andhra University, Guntur,

M. Venkataraman Esq., B.Sc., (Hons.) Research Student, Annamalai University, Annamalainagar,

A. E. Palkar Esq., B.A., Teacher, near P. O. Babujamal, Kolhapur,

Amritsagar Puri Esq., c/o Sardar Narsing Das Puri Esq., Street Paranjeet Gunj, Kapurthala,

Arjan Nath Muja Esq., B.A., Karapore-Kuski-Ranawani, Srinagar,

V. H. Ayyar Esq., B.A., Hons, Assistant, Finance Dept. New Delhi,

K. S. Ganapathy Esq., M.A., Lecturer, M. D. T. College, Tinnevelly.

The Thirteenth Conference of the Indian Mathematical Society was held at Annamalainagar at the invitation of the Annamalai University on the 28th, 29th and 30th of December 1943. The Conference was opened by M. Rutnaswami Esq., Vice-Chancellor of the University, and the Presidential Address with the title "Why Mathematics" was delivered by Dr. F. W. Levi, Hardinge Professor of Mathematics, Calcutta University, and President of the Society. About 50 papers were presented to the Conference, and in addition, 3 symposia were arranged to discuss "Partially ordered sets" "Topology" and "Modern Algebra and the theory of elementary particles." A noteworthy feature of the conference was the "Mathematics Exposition" which covered over 11 rooms and contained a large number of exhibits illustrating interesting results in mathematics and its applications to life. A detailed account of the Conference Proceedings will be published in the September-December 1943 double number of the *Student*, while the papers read at the Symposia will be issued as another double number.

The Managing Committee of the Society has accepted an endowment of Rs. 1000 from Dr. A. Narasinga Rao, for the periodical award of a medal under the following conditions:

1. The Medal shall be called "The Narasinga Rao Medal for Mathematical Research" and shall ordinarily be presented to the winner on the opening day of the Conference of the Indian Mathematical Society which meets immediately after the adjudication of the award. The Medal shall be of gold of a weight ordinarily not less than that of 1½ sovereigns and shall bear on one side the inscription "Indian Mathematical Society" and "Narasinga Rao Medal" and on the other side the name of the winner and the year of award and the place where the Conference meets.

2. The cost of the medal shall be met out of the interest earned by the "Endowment Fund" mentioned below, supplemented by such amounts as may be added to it on each occasion by the Indian Mathematical Society, the Founder or by individual donors. If this amount be found insufficient, a part of the "Endowment Fund" may be utilized for the purpose. The Endowment Fund shall be made up of:

(i) the initial endowment of Rs. 1,000 donated by the Founder, Dr. A. Narasinga Rao, Professor, Annamalai University,

(ii) such other additions as may be made to the fund by the Founder or by others, or by the Indian Mathematical Society.

The management of the Fund shall rest with the Indian Mathematical Society, which shall have the power to make such changes in the conditions of award as new circumstances may render desirable consistent with the object of the founder who shall be consulted during his life time..

3. The Medal shall be awarded to the best solution or contribution towards the solution of a specified problem in Mathematics. The problem on which theses are invited shall be announced in *The Journal of the Indian Mathematical Society, The Mathematics Student* and in the Press at least 18 months in advance of the date fixed for the submission of the theses. An Expert Committee shall be appointed from time to time by the Managing Committee of the Indian Mathematical Society to select the subject for the competition and to make arrangements for adjudicating the award, which may be divided among theses of equal merit or be withheld altogether.

4. To be eligible to compete for the medal, the competitor must be of Indian domicile and must be a member of the Indian Mathematical Society both at the time of submitting the thesis and at the time of award. There shall be no entrance fee. A person who has won the competition on one occasion shall be eligible for a second award, but shall not be eligible to compete a third time.

5. The winning thesis shall be published in one of the Society's journals and 50 free reprints shall be given to the winner, and free copies to all the Universities and other mathematical Societies in India.

The first problem has been selected and is given below:—

Problem

"It is required to characterise and study the strongest topology that can be imposed on the linear space of those real number sequences $a=(a_1, a_2, a_3, \ldots)$, that have a convergent sum $S(a)=a_1+a_2+\ldots$ to infinity, so that $S(a)$ is a continuous functional of a."

A leaflet explaining the prize problem has been sent to all members and subscribers. Copies may be had from the Editor.

K. Chandrasekharan M. Sc. and S. Minakshisundaram D. Sc. have been appointed Lecturers in the Andhra University, Guntur.

P. Kesava Menon, M. Sc., Lecturer in Mathematics Annamalai University has been appointed Lecturer in the Christian College, Tambaram.

We offer the hearty felicitations of the Indian Mathematical Society to Dr. Subramanya Chandrasekar, Professor of Theoretical Astronomy and Astrophysics at the Yerkes observatory of the Chicago University, on his recent election to the Fellowship of the Royal Society of London, he being the ninth Indian to achieve this distinction. As in the case of Ramanujan (elected at 30), this honour came early at the age of 33 and was preceeded by another rare distinction—a Fellowship of Trinity. Dr. Chandrasekhar's contributions to stellar interiors, galactic dynamics and spiral nebulae combine a rare insight into physical theories and processes with an unrivalled grasp of mathematical analysis, and are embodied in the following 3 major treatises and a large number of papers in the *Proc. Roy. Soc.*, *The Month. Notices of Royal Astron. Society* and *The Astrophysical Journal.*

(1) Introduction to the study of stellar structure, 1939, Univ. of Chicago
(2) Principles of stellar dynamics, 1942, Univ. of Chicago
(3) Stochastic Problems in Physics and Astronomy, Monograph in *Reviews of Modern Physics.*

Dr. Chandrasekhar belongs to a gifted family, being the nephew of Sir C. V. Raman. We wish him many more honours and distinctions.

We regret very much to announce the death on 28-10-1942 at the age of 47 of Prof. P. K. Kashikar of Wilson College, Bombay, an able teacher and a scholar in Samskrit and Marathi. Prof. Kashikar was very much interested in the construction of semi-regular and stellated polyhedra and has left a valuable legacy of stellated icosohedra besides other models to the Wilson College. He was the joint winner, with Prof. A. A. Krishnaswami Ayyangar of Mysore, of a prize for the following Problem proposed by V. Ramaswami Iyer, Founder of the Indian Mathematical Society: "Show that the number of ways of connecting n villages with one another by $n-1$ roads, each of them connecting two of the villages directly, is n^{n-2}. (J. I. M. S. Vol 18 Part II, p. 89, 162).

SOLUTIONS TO QUESTIONS

Question 1800

(A. A. KRISHNASWAMI AYYANGAR):—Show that

$$\int_0^x e^{-\frac{x^2}{2}}\,dx > \sqrt{\frac{\pi}{2}}\left(1-e^{-x^2}-\frac{1}{2}e^{-\frac{x^2}{2}}\right)$$

for all real x.

Solution by K. Balachandran.

Consider the function

$$\phi(x) = \int_0^x e^{-\frac{x^2}{2}}\,dx - \sqrt{\frac{\pi}{2}}\left(1-e^{-x^2}-\frac{1}{2}e^{-\frac{x^2}{2}}\right), \qquad \ldots \quad (1)$$

where x is assumed to be positive.

It is obvious $\phi(0) = \frac{1}{2}\sqrt{\frac{\pi}{2}}$, $\phi(\infty) = 0$.

Now $\phi'(x) = e^{-\frac{x^2}{2}} - \sqrt{\frac{\pi}{2}}\left(2xe^{-x^2} + \frac{x}{2}e^{-\frac{x^2}{2}}\right)$

$$= e^{-\frac{x^2}{2}}\left\{\left(1-\frac{1}{2}\sqrt{\frac{\pi}{2}}\,x\right) - \sqrt{2\pi}\,xe^{-\frac{x^2}{2}}\right\} \qquad \ldots \quad (2)$$

If we draw the straight line $y = 1 - \frac{1}{2}\sqrt{\frac{\pi}{2}}\,x$ and the curve $y = \sqrt{2\pi}\,xe^{-\frac{x^2}{2}}$, they will cut each other at just one point, say $x = x_1$, where $\phi'(x_1) = 0$. For $0 < x < x_1$, $\phi'(x)$ is positive and for $x > x_1$, it is negative.

Thus the function $\phi(x)$ in (1) increases monotonically from the value $\phi(0) = \frac{1}{2}\sqrt{\frac{\pi}{2}}$ to a maximum value $\phi(x_1)$ in the range $(0<x<x_1)$ and then decreases monotonically to the value $\phi(\infty) = 0$ for $x > x_1$. Hence for every positive x,

$$\phi(x) > 0;$$

that is, from (1)

$$\int_0^x e^{-\frac{x^2}{2}}\,dx > \sqrt{\frac{\pi}{2}}\left(1-e^{-x^2}-\frac{1}{2}e^{-\frac{x^2}{2}}\right)$$

The symmetry of these expressions shows that the restriction that x is positive can now be removed.

QUESTIONS FOR SOLUTION

1813. (A. A. KRISHNASWAMI AYYANGAR):—Find the least number of positive integral solutions $(x, y > i, j)$ possible for two simultaneous equations of the type:

$$x(x-1)(x-2)\cdots(x-i) = cy(y-1)(y-2)\cdots(y-i)$$
$$x(x-1)(x-2)\cdots(x-j) = c'y(y-1)(y-2)\cdots(y-j)$$

Discuss the curves represented by these equations.

(These curves may be called 'Lattice Curves' since they pass through the lattice points of a square. It is believed that the above equations have at most one pair of values in integers greater than i and j).

1814. (K. G. RAMANATHAN):—If $\lambda = 2\cos\frac{\pi}{5}$ and $[x]$ denotes the integral part of x, prove that

$$[\lambda^r] + [\lambda^{r-1}] + [\lambda^{r-2}] + [\lambda^{r-3}] \equiv 8 \pmod{10}$$

where r is an integer

1815. (K. G. RAMANATHAN):—If $[x]$ denotes the integral part of x and $\alpha = [\log_2 N]$, prove that

$$N - \alpha - 2 = \left[\frac{N}{2}\right] + \left[\frac{N-2}{2^2}\right] + \left[\frac{N-2^2}{2^3}\right] + \cdots + \left[\frac{N-2^{\alpha-2}}{2^{\alpha-1}}\right]$$

where N is any positive integer

1816. (K. G. RAMANATHAN):—Prove that

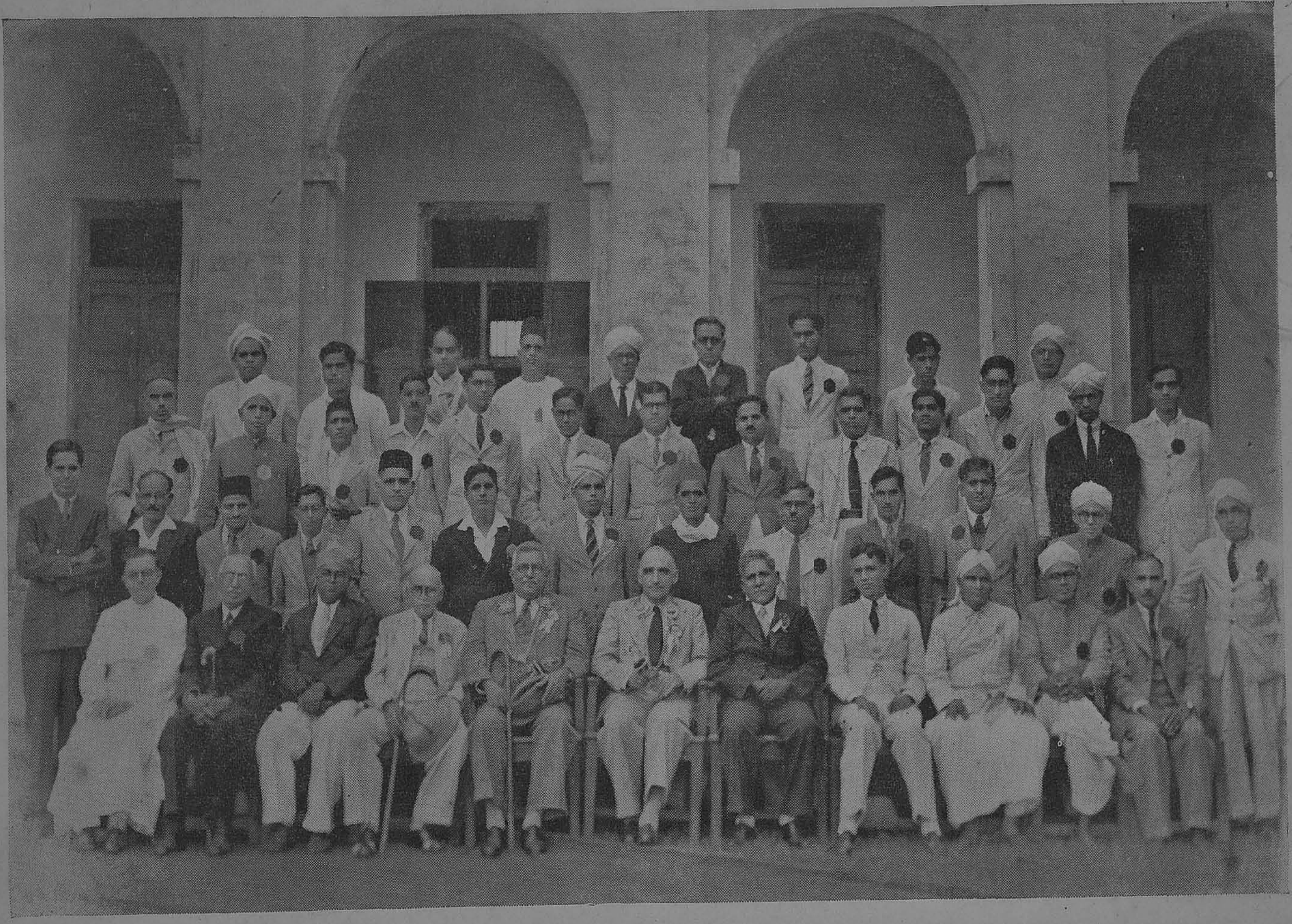
$$\frac{2^2}{(2^2-1)(3^2-1)} + \frac{2^2-3^2}{(2^2-1)(3^2-1)(5^2-1)} + \cdots = \frac{\pi^2}{6} - \frac{4}{3}$$

where 2, 3, 5, are the series of prime numbers.

1817. (P. KESAVA MENON):—If x_i $(i = 1, 2, \ldots, n)$, a, l are all positive numbers and $l^n = x_1 x_2 \ldots x_n$, prove that for all real values of k,

$$\sum_{i=1}^{n} \frac{1}{(a+x_i)^k} \lessgtr \frac{n}{(a+l)^k}$$

according as $x_i \gtrless \frac{a}{k}$ $(i = 1, 2, \ldots, n)$.



THIRTEENTH CONFERENCE OF THE INDIAN MATHEMATICAL SOCIETY

HELD AT ANNAMALAINAGAR, 28th, 29th, & 30th DECEMBER 1943.

Sitting :— Father C. Racine (*Madras*), D. D. Kapadia (*Poona*), S. Mahadevan (*Madras*), Dewan Bahadur T. Raghaviah (*Madras*), S. V. Ramamurthi (*Madras*), F. W. Levi (*Calcutta*)

M. Ruthnasawami (*Annamalainagar*), C. Ratho (*Cuddalore*), R. Vaidyanathaswami (*Madras*), G. A Srinivasan (*Madras*), S. M. Shah (*Aligarh*)

Standing First Row :— S. Subramaniam (*Delhi*), A. N. Singh (*Lucknow*), S. K. Abhyankar (*Gwalior*), S. Minakshisundaram (*Guntur*), K. R. Gunjikar (*Ahmedabad*), O. R. Suryanarayana Iyer (*Trichinopoly*)

M. V. Jambunathan (*Mysore*), L. N. Subramaniam (*Madras*), T. Venkatrayudu (*Guntur*), Afzal Ahmed (*Hyderabad*), K. Nagabhushanam (*Guntur*), R. Krishnamurthi Iyer (*Hyderabad*)

A. Narasinga Rao (*Annamalainagar*)

Standing Second Row:— K. Panchapakesa Iyer (*Pudukkottah*), S. Ramachandra Iyer (*Mangalore*), D. R. Kaprekar (*Devlali*), M. V. Subba Rao (*Madras*), Balagangadharan (*Madras*), V. A. Mahalingam (*Madras*)

T. M. Krishnmurti (*Trichinopoly*), K. Sambasiva Rao (*Guntur*), V. Ramaswami (*Guntur*), S. Sankaranarayanan (*Madras*), A. S. Puri (*Kapurthala*), T. Totadri Iyengar (*Trichinopoly*)

Suryanarayanamurti (*Madras*)

Standing Last Row :— A. A. Krishnaswami Iyengar (*Mysore*), S. Subramania Iyer (*Tirukattupalli*), K. S. Ganapati (*Tinnevelly*), O. R. Muthuswami Iyer (*Tirukattupalli*), K. Subramanian (*Tinnevelly*)

S. Raghavachari (*Kumbakonam*), K. Chandrasekharan (*Madras*), V. S. Krishnan (*Madras*), M. Venkatarama Iyer (*Mysore*)